Key to HW#3
431, Vibration CSUF spring 2009



**3.2** Calculate the solution to

$$\ddot{x}+2\dot{x}+3x=\sin t+\delta(t-\pi)$$
$$x(0)=0 \quad \dot{x}(0)=1$$

and plot the response.

**Solution:** Given: $\ddot{x}+2\dot{x}+3x=\sin t+\delta(t-\pi)$, $x(0)=0$, $\dot{x}(0)=0$

$$\omega_n=\sqrt{\frac{k}{m}}=1.732 \text{ rad/s}, \ \zeta=\frac{c}{2\sqrt{km}}=0.5774, \ \omega_d=\omega_n\sqrt{1-\zeta^2}=1.414 \text{ rad/s}$$

Total Solution:

$$x(t)=x_h+x_{p1} \qquad 0<t<\pi$$
$$x(t)=x_h+x_{p1}+x_{p2} \qquad t>\pi$$

Homogeneous: Eq. (1.36)

$$x_h(t)=Ae^{-\zeta\omega_n t}\sin(\omega_d t+\phi)=Ae^{-t}\sin(1.414t+\phi)$$

Particular: #1 (Chapter 2)

$x_{p1}(t)=X\sin(\omega t-\theta)$, where $\omega=1$ rad/s. Note that $f_0=\frac{F_0}{m}=1$

$$\Rightarrow X=\frac{f_0}{\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}}=0.3536, \text{ and } \theta=\tan^{-1}\left[\frac{2\zeta\omega_n\omega}{\omega_n^2-\omega^2}\right]=0.785 \text{ rad}$$

$$\Rightarrow x_{p1}(t)=0.3536\sin(t-0.7854)$$

Particular: #2 Equation 3.9

$$x_{p2}(t)=\frac{1}{m\omega_d}e^{-\zeta\omega_n(t-\pi)}\sin\omega_d(t-\tau)=\frac{1}{(1)(1.414)}e^{-(t-\pi)}\sin 1.414(t-\pi)$$

$$\Rightarrow x_{p2}(t)=0.7071e^{-(t-\pi)}\sin 1.414(t-\pi)$$

The total solution for $0<t<\pi$ becomes:

$$x(t)=Ae^{-t}\sin(1.414t+\phi)+0.3536\sin(t-0.7854)$$
$$\dot{x}(t)=-Ae^{-t}\sin(1.414t+\phi)+1.414Ae^{-t}\cos(1.414t+\phi)+0.3536\cos(t-0.7854)$$

$$x(0)=0=A\sin\phi-0.25\Rightarrow A=\frac{0.25}{\sin\phi}$$

$$\dot{x}(0)=1=-A\sin\phi+1.414A\cos\phi+0.25\Rightarrow 0.75=0.25-1.414(0.25)\frac{1}{\tan\phi}$$

$$\Rightarrow\phi=0.34 \text{ and } A=0.75$$

Thus for the first time interval, the response is

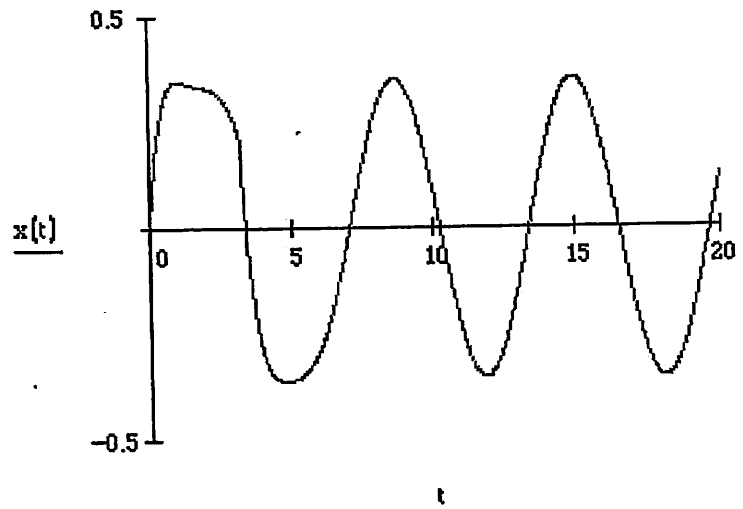
$$x(t)=0.75e^{-t}\sin(1.414t+0.34)+0.3536\sin(t-0.7854) \qquad 0<t<\pi$$

Next consider the application of the impulse at $t=\pi$:

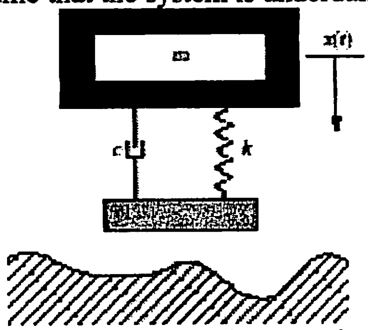
$$x(t) = x_h + x_{p1} + x_{p2}$$

$$x(t) = -0.433e^{-t}\sin(1.414t + 0.6155) + 0.3536\sin(t - 0.7854) - 0.7071e^{-(t-\pi)}\sin(1.414t - \pi) \quad t > \pi$$

The response is plotted in the following (from Mathcad):



**3.8** The vibration packages dropped from a height of $h$ meters can be approximated by considering Figure P3.8 and modeling the point of contact as an impulse applied to the system at the time of contact. Calculate the vibration of the mass $m$ after the system falls and hits the ground. Assume that the system is underdamped.




**Solution:** When the system hits the ground, it responds as if an impulse force acted on it.

From Equation (3.6): $x(t) = \dfrac{\hat{F}e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t$ where $\dfrac{\hat{F}}{m} = v_0$

Calculate $v_0$:

For falling mass: $x = \dfrac{1}{2}at^2$

So, $v_0 = gt^*$, where $t^*$ is the time of impact from height $h$

$$h = \frac{1}{2}gt^{*2} \Rightarrow t^* = \sqrt{\frac{2h}{g}}$$

$$v_0 = \sqrt{2gh}$$

Let $t = 0$ when the end of the spring hits the ground

The response is $$x(t) = \frac{\sqrt{2gh}}{\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t$$

Where $\omega_n$, $\omega_d$, and $\zeta$ are calculated from $m$, $c$, $k$. Of course the problem could be solved as a free response problem with $x_0 = 0$, $v_0 = \sqrt{2gh}$ or an impulse response with impact model as the unit velocity given.

**3.11** Calculate the response of the system

$$3\ddot{x}(t) + 6\dot{x}(t) + 12x(t) = 3\delta(t) - \delta(t-1)$$

subject to the initial conditions $x(0) = 0.01$ m and $v(0) = 1$ m/s. The units are in Newtons. Plot the response.

**Solution:** First compute the natural frequency and damping ratio:

$$\omega_n = \sqrt{\frac{12}{3}} = 2 \text{ rad/s}, \quad \zeta = \frac{6}{2\cdot 2\cdot 3} = 0.5, \quad \omega_d = 2\sqrt{1-0.5^2} = 1.73 \text{ rad/s}$$

so that the system is underdamped. Next compute the responses to the two impulses:

$$x_1(t) = \frac{\hat{F}}{m\omega_d} e^{-\zeta\omega_n t} \sin\omega_d t = \frac{3}{3(1.73)} e^{-(t-1)} \sin 1.73(t-1) = 0.577 e^{-t} \sin 1.73t, t > 0$$

$$x_2(t) = \frac{\hat{F}}{m\omega_d} e^{-\zeta\omega_n(t-1)} \sin\omega_d(t-1) = \frac{1}{3(1.73)} e^{-t} \sin 1.73t = 0.193 e^{-(t-1)} \sin 1.73(t-1), t > 1$$

Now compute the response to the initial conditions from Equation (1.36)

$$x_h(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

$$A = \sqrt{\frac{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}{\omega_d^2}}, \qquad \phi = \tan^{-1}\left[\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right] = 0.071 \text{ rad}$$

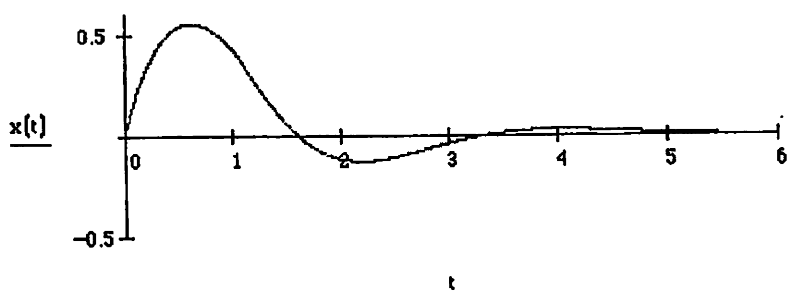
$$\Rightarrow x_h(t) = 0.5775 e^{-t} \sin(t + 0.017)$$

Using the Heaviside function the total response is

$$x(t) = 0.577 e^{-t} \sin 1.73t + 0.583 e^{-t} \sin(t + 0.017) + 0.193 e^{-(t-1)} \sin 1.73(t-1)\Phi(t-1)$$

This is plotted below in Mathcad:

$$x(t) := \left(\frac{e^{-\zeta\cdot\omega n\cdot t}}{\omega d}\cdot\sin(\omega d\cdot t) + A\cdot e^{-\zeta\cdot\omega n\cdot t}\cdot\sin(\omega d\cdot t + \phi)\right) + \left[\frac{e^{-\zeta\cdot\omega n\cdot(t-1)}}{-3\cdot\omega d}\cdot\sin[\omega d\cdot(t-1)]\right]\cdot\Phi(t-1)$$




Note the slight bump in the response at $t = 1$ when the second impact occurs.

**3.16** Calculate the response of an underdamped system to the excitation given in Figure P3.16.
Plot of a pulse input of the form $f(t) = F_0 \sin t$.

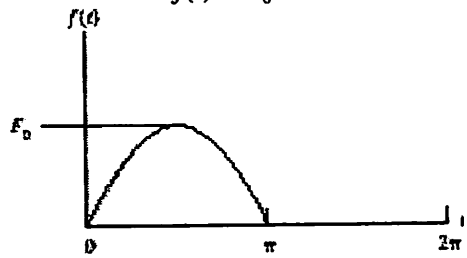



**Figure P3.16**

**Solution:**

$$x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \int_0^t \left[ F(\tau) e^{\zeta\omega_n \tau} \sin\omega_d (t-\tau) \right] d\tau$$

$$F(t) = F_0 \sin(t) \qquad t < \pi \quad \text{(From Figure P3.16)}$$

For $t \le \pi$, $$x(t) = \frac{F_0}{m\omega_d} e^{-\zeta\omega_n t} \int_0^t \left( \sin\tau e^{\zeta\omega_n \tau} \sin\omega_d (t-\tau) \right) d\tau$$

$$x(t) = \frac{F_0}{m\omega_d} e^{-\zeta\omega_n t} \times$$

$$\left[ \frac{1}{2\left[1 + 2\omega_d + \omega_n^2\right]} \left\{ e^{\zeta\omega_n t} \left[ (\omega_d - 1)\sin t - \zeta\omega_n \cos t \right] - (\omega_d - 1)\sin\omega_d t - \zeta\omega_n \cos\omega_d t \right\} \right.$$

$$\left. + \frac{1}{2\left[1 + 2\omega_d + \omega_n^2\right]} \left\{ e^{\zeta\omega_n t} \left[ (\omega_d - 1)\sin t - \zeta\omega_n \cos t \right] + (\omega_d - 1)\sin\omega_d t - \zeta\omega_n \cos\omega_d t \right\} \right]$$

For $\tau > \pi$,: $\int_0^t f(\tau)h(t-\tau)d\tau = \int_0^\pi f(\tau)h(t-\tau)d\tau + \int_\pi^t (0)h(t-\tau)d\tau$

$$x(t)=\frac{F_0}{m\omega_d}e^{-\zeta\omega_n t}\int_0^{\pi}\left(\sin\tau e^{\zeta\omega_n\tau}\sin\omega_d(t-\tau)\right)d\tau$$

$$=\frac{F_0}{m\omega_d}e^{-\zeta\omega_n t}\times$$

$$\left[\frac{1}{2\left[1+2\omega_d+\omega_n^{\ 2}\right]}\left\{\begin{matrix}e^{\zeta\omega_n t}\left[(\omega_d-1)\sin\left[\omega_d(t-\pi)\right]-\zeta\omega_n\cos\left[\omega_d(t-\pi)\right]\right]\\-(\omega_d-1)\sin\omega_d t-\zeta\omega_n\cos\omega_d t\end{matrix}\right\}\right.$$

$$\left.+\frac{1}{2\left[1+2\omega_d+\omega_n^{\ 2}\right]}\left\{\begin{matrix}e^{\zeta\omega_n t}\left[(\omega_d+1)\sin\left[\omega_d(t-\tau)\right]+\zeta\omega\cos\left[\omega_d(t-\pi)\right]\right]\\+(\omega_d-1)\sin\omega_d t-\zeta\omega_n\cos\omega_d t\end{matrix}\right\}\right]$$

Alternately, one could take a Laplace Transform approach and assume the under-damped system is a mass-spring-damper system of the form

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=F(t)$$

The forcing function given can be written as

$$F(t)=F_0\left(H(t)-H(t-\pi)\right)\sin(t)$$

Normalizing the equation of motion yields

$$\ddot{x}(t)+2\zeta\omega_n\dot{x}(t)+\omega_n^2x(t)=f_0\left(H(t)-H(t-\pi)\right)\sin(t)$$

where $f_0=\frac{F_0}{m}$ and *m*, *c* and *k* are such that $0<\zeta<1$.

Assuming initial conditions, transforming the equation of motion into the Laplace domain yields

$$X(s)=\frac{f_0\left(1+e^{-\pi s}\right)}{\left(s^2+1\right)\left(s^2+2\zeta\omega_n s+\omega_n^2\right)}$$

The above expression can be converted to partial fractions

$$X(s)=f_0\left(1+e^{-\pi s}\right)\left(\frac{As+B}{s^2+1}\right)+f_0\left(1+e^{-\pi s}\right)\left(\frac{Cs+D}{s^2+2\zeta\omega_n s+\omega_n^2}\right)$$

where *A*, *B*, *C*, and *D* are found to be

$$A = \frac{-2\zeta\omega_n}{\left(1-\omega_n^2\right)^2 + \left(2\zeta\omega_n\right)^2}$$

$$B = \frac{\omega_n^2 - 1}{\left(1-\omega_n^2\right)^2 + \left(2\zeta\omega_n\right)^2}$$

$$C = \frac{2\zeta\omega_n}{\left(1-\omega_n^2\right)^2 + \left(2\zeta\omega_n\right)^2}$$

$$D = \frac{\left(1-\omega_n^2\right) + \left(2\zeta\omega_n\right)^2}{\left(1-\omega_n^2\right)^2 + \left(2\zeta\omega_n\right)^2}$$

Notice that $X(s)$ can be written more attractively as

$$X(s) = f_0\left(\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2\zeta\omega_n s+\omega_n^2}\right) + f_0 e^{-\pi s}\left(\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2\zeta\omega_n s+\omega_n^2}\right)$$
$$= f_0\left(G(s) + e^{-\pi s}G(s)\right)$$

Performing the inverse Laplace Transform yields

$$x(t) = f_0\left(g(t) + H(t-\pi)g(t-\pi)\right)$$

where $g(t)$ is given below

$$g(t) = A\cos(t) + B\sin(t) + Ce^{-\zeta\omega_n t}\cos(\omega_d t) + \left(\frac{D - C\zeta\omega_n}{\omega_d}\right)e^{-\zeta\omega_n t}\sin(\omega_d t)$$

$\omega_d$ is the damped natural frequency, $\omega_d = \omega_n\sqrt{1-\zeta^2}$ .

Let $m$=1 kg, $c$=2 kg/sec, $k$=3 N/m, and $F_0$=2 N. The system is solved numerically. Both exact and numerical solutions are plotted below

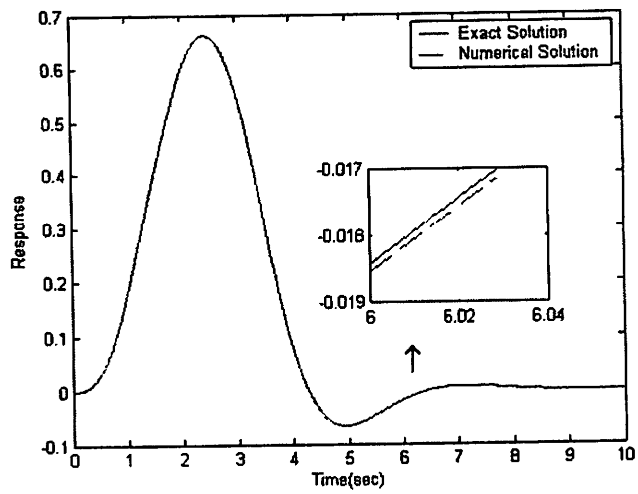



**Figure 1 Analytical vs. Numerical Solutions**

Below is the code used to solve this problem

```
% Establish a time vector
t=[0:0.001:10];

% Define the mass, spring stiffness and damping coefficient
m=1;
c=2;
k=3;

% Define the amplitude of the forcing function
F0=2;

% Calculate the natural frequency, damping ratio and normalized force amplitude
zeta=c/(2*sqrt(k*m));
wn=sqrt(k/m);
f0=F0/m;

% Calculate the damped natural frequency
wd=wn*sqrt(1-zeta^2);

% Below is the common denominator of A, B, C and D (partial fractions
% coefficients)
dummy=(1-wn^2)^2+(2*zeta*wn)^2;

% Hence, A, B, C, and D are given by
A=-2*zeta*wn/dummy;
B=(wn^2-1)/dummy;
C=2*zeta*wn/dummy;
```

```
D=((1-wn^2)+(2*zeta*wn)^2)/dummy;

% EXACT SOLUTION
%
***********************************************************************
*
%
***********************************************************************
*
for i=1:length(t)
   % Start by defining the function g(t)
   g(i)=A*cos(t(i))+B*sin(t(i))+C*exp(-zeta*wn*t(i))*cos(wd*t(i))+((D-
C*zeta*wn)/wd)*exp(-zeta*wn*t(i))*sin(wd*t(i));
   % Before t=pi, the response will be only g(t)
   if t(i)<pi
      xe(i)=f0*g(i);
      % d is the index of delay that will correspond to t=pi
      d=i;
   else
      % After t=pi, the response is g(t) plus a delayed g(t). The amount
      % of delay is pi seconds, and it is d increments
      xe(i)=f0*(g(i)+g(i-d));
   end;
end;

% NUMERICAL SOLUTION
%
***********************************************************************
*
%
***********************************************************************
*

% Start by defining the forcing function
for i=1:length(t)
   if t(i)<pi
      f(i)=f0*sin(t(i));
   else
      f(i)=0;
   end;
end;

% Define the transfer functions of the system
% This is given below
%       1
% --------------------------
```

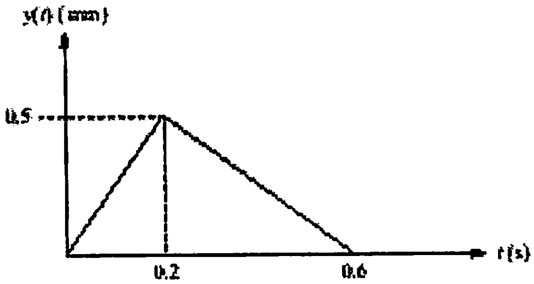
```
% s^2+2*zeta*wn+wn^2

% Define the numerator and denominator
num=[1];
den=[1 2*zeta*wn wn^2];
% Establish the transfer function
sys=tf(num,den);

% Obtain the solution using lsim
xn=lsim(sys,f,t);

% Plot the results
figure;
set(gcf,'Color','White');
plot(t,xe,t,xn,'--');
xlabel('Time(sec)');
ylabel('Response');
legend('Forcing Function','Exact Solution','Numerical Solution');
text(6,0.05,'\uparrow','FontSize',18);
axes('Position',[0.55 0.3/0.8 0.25 0.25])
plot(t(6001:6030),xe(6001:6030),t(6001:6030),xn(6001:6030),'--');
```

**3. 21** A machine resting on an elastic support can be modeled as a single-degree-of-freedom, spring-mass system arranged in the vertical direction. The ground is subject to a motion $y(t)$ of the form illustrated in Figure P3.221. The machine has a mass of 5000 kg and the support has stiffness $1.5\text{x}10^3$ N/m. Calculate the resulting vibration of the machine.




**Solution:** Given $m$ = 5000 kg, $k = 1.5\text{x}10^3$ N/m, $\omega_n = \sqrt{k/m} = 0.548$ rad/s and that the ground motion is given by:

$$y(t) = \begin{cases} 2.5t & 0 \le t \le 0.2 \\ 0.75 - 1.25t & 0.2 \le t \le 0.6 \\ 0 & t \ge 0.6 \end{cases}$$

The equation of motion is $m\ddot{x} + k(x - y) = 0$ or $m\ddot{x} + kx = ky = F(t)$ The impulse response function computed from equation (3.12) for an undamped system is

$$h(t-\tau) = \frac{1}{m\omega_n}\sin\omega_n(t-\tau)$$

This gives the solution by integrating a $yh$ across each time step:

$$x(t) = \frac{1}{m\omega_n}\int_0^t ky(\tau)\sin\omega_n(t-\tau)d\tau = \omega_n\int_0^t y(\tau)\sin\omega_n(t-\tau)d\tau$$

For the interval $0 \le t \le 0.2$:

$$x(t) = \omega_n\int_0^t 2.5\tau\sin\omega_n(t-\tau)d\tau$$

$$\Rightarrow \underline{x(t) = 2.5t - 4.56\sin 0.548t \text{ mm } \ 0 \le t \le 0.2}$$

For the interval $0.2 \le t \le 0.6$:

$$x(t) = \omega_n\int_0^{0.2} 2.5\tau\sin\omega_n(t-\tau)d\tau + \omega_n\int_{0.2}^t (0.75 - 1.25\tau)\sin\omega_n(t-\tau)d\tau$$

$$= 0.75 - 0.5\cos 0.548(t-0.2) - 1.25t + 2.28\sin 0.548(t-0.2)$$

Combining this with the solution from the first interval yields:

$$\underline{x(t) = 0.75 + 1.25t - 0.5\cos 0.548(t-0.2)}$$

$$\underline{+6.48\sin 0.548(t-0.2) - 4.56\sin 0.548(t-0.2) \text{ mm } \ 0.2 \le t \le 0.6}$$

Finally for the interval $t \ge 0.6$:

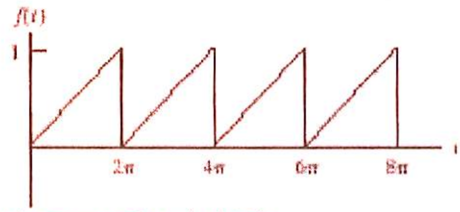
$$x(t) = \omega_n \int_0^{0.2} 2.5t \sin\omega_n(t-\tau)d\tau + \omega_n \int_{0.2}^{0.6} (0.75 - 1.25t)\sin\omega_n(t-\tau)d\tau + \omega_n \int_0^{t} (0)\sin\omega_n(t-\tau)d\tau$$

$$= -0.5\cos 0.548(t-0.2) - 2.28\sin 0.548(t-0.6) + 2.28\sin 0.548(t-0.2)$$

Combining this with the total solution from the previous time interval yields:

$$x(t) = -0.5\cos 0.548(t-0.2) + 6.84\sin 0.548(t-0.2) - 2.28\sin 0.548(t-0.6)$$
$$-4.56\sin 0.548t \quad \text{mm } t \geq 0.6$$

**3.29** Determine the Fourier series representation of the sawtooth curve illustrated in Figure P3.29.




**Solution:** The sawtooth curve of period $T$ is

$$F(t)=\frac{1}{2\pi}t \qquad 0\le t\le 2\pi$$

Determine coefficients $a_0, a_n, b_n$ :

$$a_0=\frac{2}{T}\int_0^T F(t)dt=\frac{2}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}t\right)dt=\left(\frac{1}{2\pi^2}\right)\frac{1}{2}t^2\Big|_0^{2\pi}$$

$$=\frac{1}{4\pi^2}\left[4\pi^2-0\right]=1$$

$$a_n=\frac{2}{T}\int_0^T F(t)\cos n\omega_T t dt, \text{ where } \omega_T=\frac{2\pi}{T}=\frac{2\pi}{2\pi}=1$$

$$=\frac{2}{2\pi}\left[\int_0^{2\pi}\left(\frac{1}{2\pi}t\right)\cos nt dt\right]=\frac{1}{2\pi^2}\left[\int_0^{2\pi}t\cos nt dt\right]$$

$$=\frac{1}{2\pi^2}\left[\frac{1}{n^2}\cos nt+\frac{1}{n}t\sin nt\right]_0^{2\pi}=\frac{1}{2\pi^2}\left[\frac{1}{n^2}(1-1)+\frac{1}{n}(0-0)\right]=0$$

$$b_n=\frac{2}{T}\int_0^T F(t)\sin n\omega_T t dt=\frac{2}{2\pi}\left[\int_0^{2\pi}\left(\frac{1}{2\pi}t\right)\sin nt dt\right]=\frac{1}{2\pi^2}\left[\int_0^{2\pi}t\sin nt dt\right]$$

$$=\frac{1}{2\pi^2}\left[\frac{1}{n^2}\sin nt-\frac{1}{n}t\cos nt\right]_0^{2\pi}=\frac{1}{2\pi^2}\left[\frac{1}{n^2}(0-0)-\frac{1}{n}(2\pi-0)\right]$$

$$=\frac{1}{2\pi^2}\left(\frac{-2\pi}{n}\right)=\frac{-1}{\pi n}$$

Fourier Series

$$F(t)=\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{-1}{\pi n}\right)\sin nt$$

$$F(t)=\frac{1}{2}-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin nt$$

**3.38** Solve the following system for the response $x(t)$ using Laplace transforms:

$$100\ddot{x}(t) + 2000x(t) = 50\delta(t)$$

where the units are in Newtons and the initial conditions are both zero.

**Solution:**

First divide by the mass to get

$$\ddot{x} + 20x(t) = 0.5\delta(t)$$

Take the Laplace Transform to get

$$(s^2 + 20)X(s) = 0.5$$

So

$$X(s) = \frac{0.5}{s^2 + 20}$$

Taking the inverse Laplace Transform using entry 5 of Table 3.1 yields

$$X(s) = \frac{0.5}{\sqrt{20}} \cdot \frac{\omega}{s^2 + \omega^2} \quad \text{where } \omega = \sqrt{20}$$

$$\Rightarrow x(t) = \frac{1}{4\sqrt{5}} \sin\sqrt{20}t$$

**3.44** Calculate the response spectrum of an undamped system to the forcing function

$$F(t)=\begin{cases} F_0 \sin\dfrac{\pi t}{t_1} & 0 \le t \le t_1 \\ 0 & t > t_1 \end{cases}$$

assuming the initial conditions are zero.

**Solution:** Let $\omega = \pi / t_1$. The solution is the homogeneous solution $x_h(t)$ and the particular solution $x_p(t)$ or $x(t) = x_h(t) + x_p(t)$. Thus

$$x(t) = A\cos\omega_n t + B\sin\omega_n t + \left(\frac{F_0}{k - m\omega^2}\right)\sin\omega t$$

where $A$ and $B$ are constants and $\omega_n$ is the natural frequency of the system:
Using the initial conditions $x(0) = \dot{x}(0) = 0$ the constants $A$ and $B$ are

$$A = 0, \quad B = \frac{-F_0\omega}{\omega_n\left(k - m\omega^2\right)}$$

so that $x(t) = \dfrac{F_0 / k}{1-(\omega/\omega_n)^2}\left\{\sin\omega t - \dfrac{\omega}{\omega_n}\sin\omega_n t\right\},\ 0 \le t \le t_1$

Which can be written as (where $\delta = F_0 / k$ the static deflection)

$$\frac{x(t)}{\delta} = \frac{1}{1-\left(\dfrac{\tau}{2t_1}\right)^2}\left\{\sin\frac{\pi t}{t_1} - \frac{\tau}{2t_1}\sin\frac{2\pi t}{\tau}\right\},\ 0 \le t \le t_1$$

and where $\tau = 2\pi / \omega_n$. After $t_1$ the solution is a free response

$$x(t) = A'\cos\omega_n t + B'\sin\omega_n t,\ t > t_1$$

where the constants $A'$ and $B'$ can be found by using the values of $x(t = t_1)$ and $\dot{x}(t = t_1),\ t > t_0$. This gives

$$x(t = t_1) = a\left[-\frac{\tau}{2t_1}\sin\frac{2\pi t_1}{\tau}\right] = A'\cos\omega_n t_1 + B'\sin\omega_n t_1$$

$$\dot{x}(t = t_1) = a\left\{\frac{\pi}{t_1} - \frac{\pi}{t_1}\cos\frac{2\pi t_1}{\tau}\right\} = -\omega_n A'\sin\omega_n t_1 + \omega_n B'\cos\omega_n t$$

where

$$a = \frac{\delta}{1-\left(\dfrac{\tau}{2t_1}\right)^2}$$

These are solved to yield

$$A' = \frac{a\pi}{\omega_n t_1}\sin\omega_n t_1, \quad B' = -\frac{a\pi}{\omega_n t_1}\left[1+\cos\omega_n t_1\right]$$

So that after $t_1$ the solution is

$$\frac{x(t)}{\delta} = \frac{(\tau/t_1)}{2\left\{1-(\tau/2t_1)^2\right\}}\left[\sin 2\pi\left(\frac{t_1}{\tau}-\frac{t}{\tau}\right)-\sin 2\pi\frac{t}{\tau}\right], t \geq t_1$$

**3.50** Calculate the frequency response function for the compliance of Problem 3.49.

**Solution:** From problem 3.49,

$$H(s)=\frac{1}{as^4+bs^3+cs^2+ds+e}$$

Substitute $s=j\omega$ to get the frequency response function:

$$H(j\omega)=\frac{1}{a(j\omega)^4+b(j\omega)^3+c(j\omega)^2+d(j\omega)+e}$$

$$H(j\omega)=\frac{a\omega^4-c\omega^2+e-j(-b\omega^3+d\omega)}{(a\omega^4-c\omega^2+e)^2+(-b\omega^3+d\omega)^2}$$

**3.49** Calculate the compliance transfer function for a system described by

$$a\ddddot{x}+b\dddot{x}+c\ddot{x}+d\dot{x}+ex=f(t)$$

where $f(t)$ is the input force and $x(t)$ is a displacement.

**Solution:**

The compliance transfer function is $\dfrac{X(s)}{F(s)}$.

Taking the Laplace Transform yields

$$(as^4+bs^3+cs^2+ds+e)X(s)=F(s)$$

So, $\dfrac{X(s)}{F(s)}=\dfrac{1}{as^4+bs^3+cs^2+ds+e}$