HW # 1 Key

1.6 Find the equation of motion for the system of Figure P1.6, and find the natural frequency. In particular, using static equilibrium along with Newton's law, determine what effect gravity has on the equation of motion and the system's natural frequency. Assume the block slides without friction.

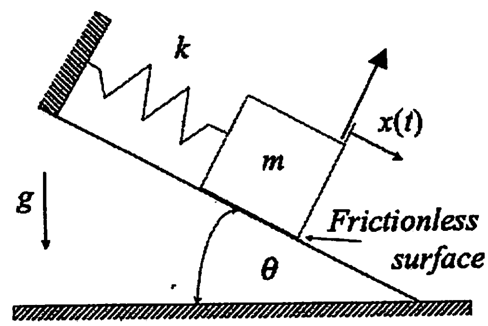
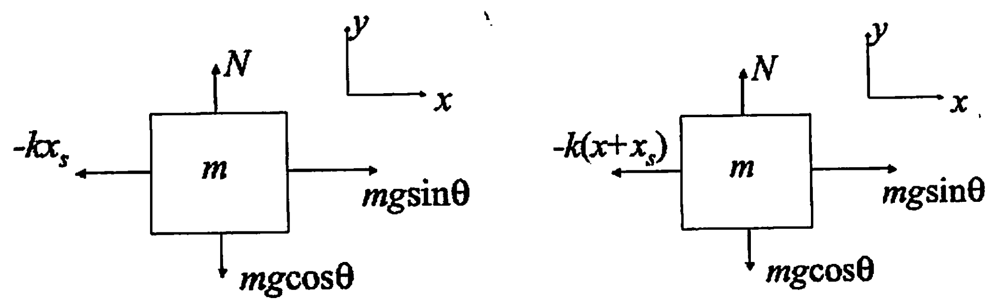


Figure P1.6

Solution:
Choosing a coordinate system along the plane with positive down the plane, the free-body diagram of the system for the static case is given and (a) and for the dynamic case in (b):



In the figures, N is the normal force and the components of gravity are determined by the angle θ as indicated. From the static equilibrium: $-kx_s + mg\sin\theta = 0$. Summing forces in (b) yields:

$$\sum F_i = m\ddot{x}(t) \Rightarrow m\ddot{x}(t) = -k(x + x_s) + mg\sin\theta$$
$$\Rightarrow m\ddot{x}(t) + kx = -kx_s + mg\sin\theta = 0$$
$$\Rightarrow \underline{m\ddot{x}(t) + kx = 0}$$
$$\Rightarrow \underline{\omega_n = \sqrt{\frac{k}{m}}} \text{ rad/s}$$

1.16 A machine part is modeled as a pendulum connected to a spring as illustrated in Figure P1.16. Ignore the mass of pendulum's rod and derive the equation of motion. Then following the procedure used in Example 1.1.1, linearize the equation of motion and compute the formula for the natural frequency. Assume that the rotation is small enough so that the spring only deflects horizontally.

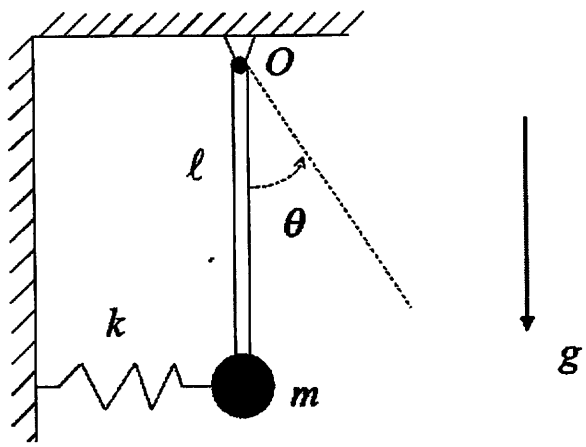
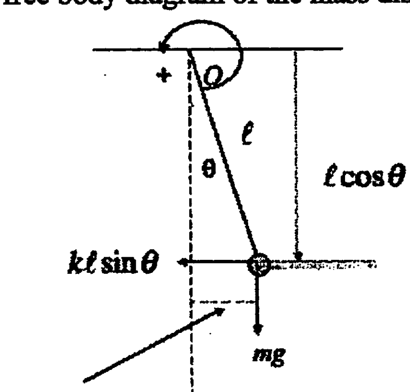


Figure P1.16

Solution: Consider the free body diagram of the mass displaced from equilibrium:



There are two forces acting on the system to consider, if we take moments about point O (then we can ignore any forces at O). This yields

$$\sum M_O = J_O\alpha \Rightarrow m\ell^2\ddot{\theta} = -mg\ell\sin\theta - k\ell\sin\theta \bullet \ell\cos\theta$$

$$\Rightarrow \underline{m\ell^2\ddot{\theta} + mg\ell\sin\theta + k\ell^2\sin\theta\cos\theta = 0}$$

Next consider the small θ approximations to that $\sin\theta \sim \theta$ and $\cos\theta = 1$. Then the linearized equation of motion becomes:

$$\ddot{\theta}(t) + \left(\frac{mg + k\ell}{m\ell}\right)\theta(t) = 0$$

Thus the natural frequency is

$$\underline{\omega_n = \sqrt{\frac{mg + k\ell}{m\ell}} \text{ rad/s}}$$

1.32 Solve $\ddot{x}+2\dot{x}+2x=0$ for $x_0 = 0$ mm, $v_0 = 1$ mm/s and sketch the response. You may wish to sketch *x(t)* = e^{-t} and *x(t)* =-e^{-t} first.

Solution:

Given $\ddot{x}+2\dot{x}+x=0$ where $x_0 = 0$, $v_0 = 1$ mm/s

Let: $x = ae^{rt} \Rightarrow \dot{x} = are^{rt} \Rightarrow \ddot{x} = ar^2e^{rt}$

Substitute into the equation of motion to get

$$ar^2e^{rt} + 2are^{rt} + ae^{rt} = 0 \Rightarrow r^2 + 2r + 1 = 0 \Rightarrow r_{1,2} = -1 \pm i$$

So

$$x = c_1e^{(-1+i)\,t} + c_2e^{(-1-i)\,t} \Rightarrow \dot{x} = (-1+i)c_1e^{(-1+i)\,t} + (-1-i)c_2e^{(-1-i)\,t}$$

Initial conditions:

$$x_0 = x(0) = c_1 + c_2 = 0 \quad \Rightarrow \quad c_2 = -c_1 \qquad (1)$$
$$v_0 = \dot{x}(0) = (-1+i)c_1 + (-1-i)c_2 = 1 \qquad (2)$$

Substituting equation (1) into (2)

$$v_0 = (-1+i)c_1 - (-1-i)c_1 = 1$$
$$c_1 = -\frac{1}{2}i, \; c_2 = \frac{1}{2}i$$
$$x(t) = -\frac{1}{2}ie^{(-1+i)\,t} + \frac{1}{2}ie^{(-1-i)\,t} = -\frac{1}{2}ie^{-t}\left(e^{it} - e^{-it}\right)$$

Applying Euler's formula

$$x(t) = -\frac{1}{2}ie^{-t}(\cos t + i\sin t - (\cos t - i\sin t))$$
$$\underline{x(t) = e^{-t}\sin t}$$

Alternately use equations (1.36) and (1.38). The plot is similar to figure 1.11.

1.43 Solve $\ddot{x} - \dot{x} + x = 0$ with $x_0 = 1$ and $v_0 = 0$ for $x(t)$ and sketch the response.

Solution: This is a problem with negative damping which can be used to tie into Section 1.8 on stability, or can be used to practice the method for deriving the solution using the method suggested following equation (1.13) and eluded to at the start of the section on damping. To this end let $x(t) = Ae^{\lambda t}$ the equation of motion to get:

$$(\lambda^2 - \lambda + 1)e^{\lambda t} = 0$$

This yields the characteristic equation:

$$\lambda^2 - \lambda + 1 = 0 \Rightarrow \lambda = \frac{1}{2} \pm \frac{\sqrt{3}}{2} j, \quad \text{where } j = \sqrt{-1}$$

There are thus two solutions as expected and these combine to form

$$x(t) = e^{0.5t}(Ae^{\frac{\sqrt{3}}{2}jt} + Be^{-\frac{\sqrt{3}}{2}jt})$$

Using the Euler relationship for the term in parenthesis as given in Window 1.4, this can be written as

$$x(t) = e^{0.5t}(A_1 \cos\frac{\sqrt{3}}{2}t + A_2 \sin\frac{\sqrt{3}}{2}t)$$

Next apply the initial conditions to determine the two constants of integration:

$$x(0) = 1 = A_1(1) + A_2(0) \Rightarrow A_1 = 1$$

Differentiate the solution to get the velocity and then apply the initial velocity condition to get

$$\dot{x}(t) =$$

$$\frac{1}{2}e^0(A_1 \cos\frac{\sqrt{3}}{2}0 + A_2 \sin\frac{\sqrt{3}}{2}0) + e^0\frac{\sqrt{3}}{2}(-A_1 \sin\frac{\sqrt{3}}{2}0 + A_2 \cos\frac{\sqrt{3}}{2}0) = 0$$

$$\Rightarrow A_1 + \sqrt{3}(A_2) = 0 \Rightarrow A_2 = -\frac{1}{\sqrt{3}},$$

$$\Rightarrow x(t) = e^{0.5t}(\cos\frac{\sqrt{3}}{2}t - \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t)$$

This function oscillates with increasing amplitude as shown in the following plot which shows the increasing amplitude. This type of response is referred to as a flutter instability. This plot is from Mathcad.

1.62 Use Lagrange's formulation to calculate the equation of motion and the natural frequency of the system of Figure P1.62. Model each of the brackets as a spring of stiffness k, and assume the inertia of the pulleys is negligible.

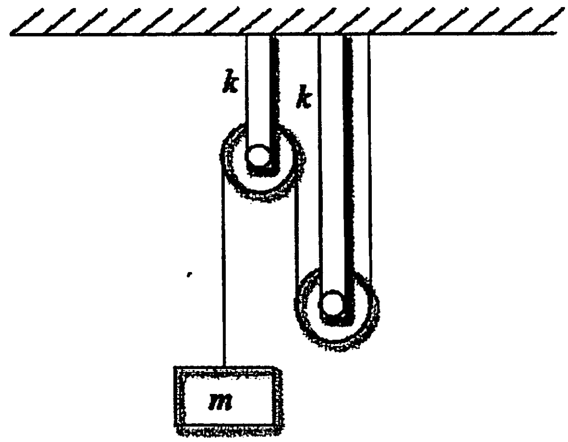


Figure P1.62

Solution: Let x denote the distance mass m moves, then each spring will deflects a distance $x/4$. Thus the potential energy of the springs is

$$U = 2 \times \frac{1}{2} k \left(\frac{x}{4}\right)^2 = \frac{k}{16} x^2$$

The kinetic energy of the mass is

$$T = \frac{1}{2} m\dot{x}^2$$

Using the Lagrange formulation in the form of Equation (1.64):

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2} m\dot{x}^2\right)\right) + \frac{\partial}{\partial x}\left(\frac{kx^2}{16}\right) = 0 \Rightarrow \frac{d}{dt}\left(m\dot{x}\right) + \frac{k}{8} x = 0$$

$$\Rightarrow \underline{m\ddot{x} + \frac{k}{8} x = 0} \Rightarrow \underline{\omega_n = \frac{1}{2}\sqrt{\frac{k}{2m}}} \text{ rad/s}$$

1.64 Lagrange's formulation can also be used for non-conservative systems by adding the applied non-conservative term to the right side of equation (1.64) to get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial U}{\partial q_i}+\frac{\partial R_i}{\partial \dot{q}_i}=0$$

Here R_i is the *Rayleigh dissipation function* defined in the case of a viscous damper attached to ground by

$$R_i=\frac{1}{2}c\dot{q}_i^2$$

Use this extended Lagrange formulation to derive the equation of motion of the damped automobile suspension of Figure P1.64

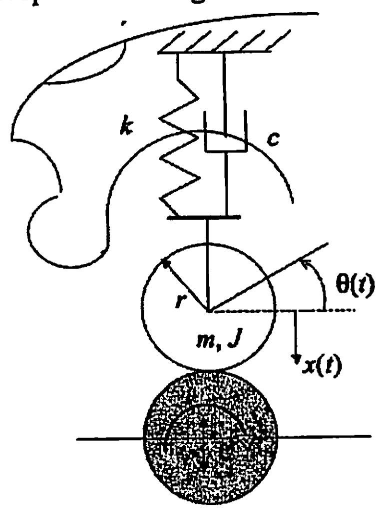


Figure P1.64

Solution: The kinetic energy is (see Example 1.4.1):

$$T=\frac{1}{2}(m+\frac{J}{r^2})\dot{x}^2$$

The potential energy is:

$$U=\frac{1}{2}kx^2$$

The Rayleigh dissipation function is

$$R=\frac{1}{2}c\dot{x}^2$$

The Lagrange formulation with damping becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial U}{\partial q_i}+\frac{\partial R_i}{\partial \dot{q}_i}=0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}(m+\frac{J}{r^2})\dot{x}^2\right)\right)+\frac{\partial}{\partial x}\left(\frac{1}{2}kx^2\right)+\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}c\dot{x}^2\right)=0$$

$$\Rightarrow (m+\frac{J}{r^2})\ddot{x}+c\dot{x}+kx=0$$

Problems and Solutions Section 1.8 (1.90 through 1.93)

1.90 Consider the system of Figure 1.90 and (a) write the equations of motion in terms of the angle, θ, the bar makes with the vertical. Assume linear deflections of the springs and linearize the equations of motion. Then (b) discuss the stability of the linear system's solutions in terms of the physical constants, m, k, and ℓ. Assume the mass of the rod acts at the center as indicated in the figure.

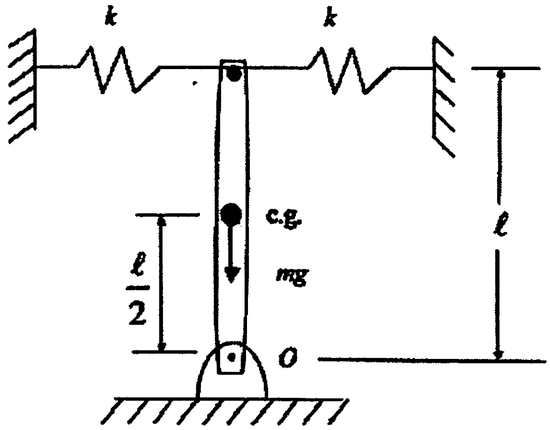


Figure P1.90

Solution: Note that from the geometry, the springs deflect a distance $kx = k(\ell\sin\theta)$ and the cg moves a distance $\ell/2\cos\theta$. Thus the total potential energy is

$$U = 2\times\frac{1}{2}k(\ell\sin\theta)^2 - \frac{mg\ell}{2}\cos\theta$$

wrong

$$mg\left(\frac{\ell}{2}-\frac{\ell}{2}\cos\theta\right) = -mg\frac{\ell}{2}\cos\theta$$

and the total kinetic energy is

$$T = \frac{1}{2}J_O\dot{\theta}^2 = \frac{1}{2}\frac{m\ell^2}{3}\dot{\theta}^2$$

The Lagrange equation (1.64) becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right)+\frac{\partial U}{\partial\theta} = \frac{d}{dt}\left(\frac{m\ell^2}{3}\dot{\theta}\right)+2k\ell\sin\theta\cos\theta-\frac{1}{2}mg\ell\sin\theta = 0$$

Using the linear, small angle approximations $\sin\theta\approx\theta$ and $\cos\theta\approx 1$ yields

$$\text{a)}\quad \frac{m\ell^2}{3}\ddot{\theta}+\left(2k\ell^2-\frac{mg\ell}{2}\right)\theta = 0$$

Since the leading coefficient is positive the sign of the coefficient of θ determines the stability.

b)

if $2k\ell-\frac{mg}{2}>0\Rightarrow 4k>\frac{mg}{\ell}\Rightarrow$ the system is stable

if $4k = mg\Rightarrow\theta(t) = at+b\Rightarrow$ the system is unstable

if $2k\ell-\frac{mg}{2}<0\Rightarrow 4k<\frac{mg}{\ell}\Rightarrow$ the system is unstable

Note that physically this results states that the system's response is stable as long as the spring stiffness is large enough to over come the force of gravity.