Handout Monday 4/28/2008.

Chapter 10: The Kolmogorov Equations

The purpose of this note is to develop the forward Kolmogorov equations, and also the backward Kolmogorov equations, for the case of a pure-jump, continuous-time Markov chain.

For a stationary Markov chain, define the transition probability

$$p_{ij}(t) = P(X(\tau + t) = j \mid X(\tau) = i) .$$

Denote the state space by I. Then, considering the transition probability $p_{ij}(t+s)$, and partitioning over all intermediate states $k \in I$ at time s, it follows that

$$p_{ij}(t+s) = \sum_{k \in I} p_{ik}(s) p_{kj}(t) .$$

These equations are the Chapman-Kolmogorov equations, which play a fundamental role throughout the analysis of Markov chains. From these equations, we can derive the forward Kolmogorov equations and the backward Kolmogorov equations.

The Forward Kolmogorov Equations We have

$$p_{ij}(t+s) = \sum_{k \in I} p_{ik}(t) p_{kj}(s) = \sum_{k \neq j} p_{ik}(t) p_{kj}(s) + p_{ij}(t) p_{jj}(s) .$$

Thus,

$$p_{ij}(t+s) - p_{ij}(t) = \sum_{k \neq j} p_{ik}(t) p_{kj}(s) + p_{ij}(t)(p_{jj}(s) - 1) .$$

Dividing both sides by s, and proceeding formally, we can take the limit as $s \to 0^+$, to get

$$p'_{ij}(t) = \sum_{k \neq j} p_{ik}(t) q_{kj} - v_j p_{ij}(t) .$$

These equations are called the forward Kolmogorov equations.

The Backward Kolmogorov Equations We have

$$p_{ij}(t+s) = \sum_{k \in I} p_{ik}(s) p_{kj}(t) = \sum_{k \neq i} p_{ik}(s) p_{kj}(t) + p_{ii}(s) p_{ij}(t) \ .$$

Thus,

$$p_{ij}(t+s) - p_{ij}(t) = \sum_{k \in I} p_{ik}(s) p_{kj}(t) = \sum_{k \neq i} p_{ik}(s) p_{kj}(t) + (p_{ii}(s) - 1) p_{ij}(t) \ .$$

Dividing both sides by s, and proceeding formally, we can take the limit as $s \to 0^+$, to get

$$p'_{ij}(t) = \sum_{k \neq j} q_{ik} p_{kj}(t) - v_i p_{ij}(t) \ .$$

These equations are called the backward Kolmogorov equations.

Let $P(t)$ be the matrix whose (i,j)-th entry is $p_{ij}(t)$. Denote the state probability vector at time t by $z(t)$. Thus, $z_n(t) = P(X(t) = n)$, for $n \in I$. For any time $t \geq 0$, we then have $z(t) = z(0)P(t)$. Next, define Q to be the matrix whose (i,j)-th entry is q_{ij}, for $i \neq j$, and $q_{ii} = -v_i$. Then the forward Kolmogorov equations can be written

$$P'(t) = P(t)Q \ , \quad \text{for} \quad t > 0 \ ,$$

while the backward Kolmogorov equations can be written

$$P'(t) = QP(t) \ , \quad \text{for} \quad t > 0 \ .$$

In developing mathematical models using continuous time Markov chains, the elements of the matrix Q are typically determined first. See for instance, Example 10.3.1 in the text, and also Problems 10.4, 10.5, and 10.6. Then, in theory at least, the differential equations above can be solved to find the transition matrix $P(t)$ and the state probability vector $z(t)$ defined above.

Example 1 The Poisson process: The Poisson process, with rate constant $\lambda > 0$, is a pure birth process with state space $I = \{0, 1, 2, \cdots\}$, for which $q_{i,i+1} = \lambda$, $v_i = \lambda$, and $q_{i,j} = 0$ otherwise. Thus, the matrix Q has the form

$$Q = \begin{bmatrix} -\lambda & \lambda & 0 & \cdots \\ 0 & -\lambda & \lambda & \cdots \\ 0 & 0 & -\lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} .$$

The entries in the transition matrix $P(t)$ can be found by induction. However, we will just find the state probability vector $z(t)$, assuming the process starts in state 0. Recalling that $z_n(t) = P(X(t) = n)$, we have $z(0) = (1, 0, 0, \cdots)$. In effect, we are finding the first row of $P(t)$, since at any time $t \geq 0$, $z(t) = z(0)P(t)$. Using the forward Kolmogorov equations gives us

$$z'(t) = z(0)P'(t) = z(0)P(t)Q = z(t)Q .$$

Thus, for $n = 0$, we have $z_0'(t) = -\lambda z_0(t)$, and for $n \geq 1$,

$$z_n'(t) = \lambda z_{n-1}(t) - \lambda z_n(t) .$$

The equation for $n = 0$ yields $z_0(t) = e^{-\lambda t}$, where we have used the initial condition $z_0(0) = 1$. Next, for $n \geq 1$, using the integrating factor $e^{\lambda t}$, we can solve for $z_n(t)$ to get

$$z_n(t) = \lambda e^{-\lambda t} \int_0^t e^{\lambda s} z_{n-1}(s) ds ,$$

where we have invoked the initial condition $z_n(0) = 0$ for $n \geq 1$. Working with these equations successively, starting with z_0, gives us

$$z_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t} , \quad \text{for} \quad n \geq 0 . \qquad \blacksquare$$

another derivation for poisson process

Example 2 A device is either operational (state 1), or being repaired (state 0). If it is in state 1, it can fail in an interval of time $(t, t+h)$ with probability $\mu h + o(h)$. If it is in state 0, it can be repaired and become operational in an interval of time $(t, t+h)$ with probability $\lambda h + o(h)$. For this model, we have

$$Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix} .$$

try to solve use
Z vector.

The forward Kolmogorov equations $P'(t) = P(t)Q$ are, in the first row:

$$p'_{0,0}(t) = -\lambda p_{0,0}(t) + \mu p_{0,1}(t) \quad \text{and} \quad p'_{0,1}(t) = \lambda p_{0,0}(t) - \mu p_{0,1}(t) ,$$

4 equations
4 unknowns

and in the second row:

$$p'_{1,0}(t) = -\lambda p_{1,0}(t) + \mu p_{1,1}(t) \quad \text{and} \quad p'_{1,1}(t) = \lambda p_{1,0}(t) - \mu p_{1,1}(t) .$$

From the first set of equations, noting that $p_{0,0}(t) + p_{0,1}(t) = 1$, we obtain the single differential equation

$$p'_{0,1}(t) + (\lambda + \mu) p_{0,1}(t) = +\lambda .$$

one diff. equation for $P'_{0,1}$

Using the initial condition $p_{0,1}(0) = 0$, and employing the integrating factor $e^{-(\lambda+\mu)t}$, yields

$$p_{0,1} = \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} ,$$

and thus,

$$p_{0,0} = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} .$$

In a similar way, it follows that

$$p_{1,0} = \frac{\mu}{\lambda+\mu} - \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t} ,$$

and,

$$p_{1,1} = \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t} .$$

Note that by taking the limit as $t \to \infty$, the long-run state probability vector π is found to be

$$\pi = \left(\frac{\mu}{\lambda+\mu} , \frac{\lambda}{\lambda+\mu} \right) .$$

This result could also have been obtained by solving the balance equations $\pi Q = 0$, subject to $\pi_0 + \pi_1 = 1$. ■

Exercises

1. For the machine repair problem in Problem 10.4 of the notes, what is the Q matrix? Assume there is only one repair person ($s = 1$).

2. For the machine repair problem with spares in Problem 10.5 of the notes, what is the Q matrix? Assume there is one repair person ($s = 1$), and one machine ($m = 1$).

3. For the light bulb problem, Problem 10.6 of the notes, what is the Q matrix?