PRACTISE PROBLEMS FOR SOLVING FIRST ORDER PDEs USING METHOD OF CHARACTERISTICS

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1 Problem 2

Solve

$$u_t + (xt)\, u_x = 0 \tag{1}$$

$u(x,0) = 2x$

Solution

Seek solution where $u(s) = u(t(s), x(s)) =$ constant,hence

$$\frac{du}{ds} = \frac{\partial u}{\partial t}\frac{dt}{ds} + \frac{\partial u}{\partial x}\frac{dx}{ds} = 0$$

Compare to (1) we see that $\frac{dt}{ds} = 1$ or $t = s$ and $\frac{dx}{ds} = xt$, but since $t = s$ then $\frac{dx}{ds} = xs$, and this has solution $x = x_0 \exp\left(\frac{s^2}{2}\right)$ but $s = t$, hence

$$x = x_0 \exp\left(\frac{t^2}{2}\right) \tag{2}$$

Now at $t = 0$, the solution is $2x_0$, but this solution is valid any where on this characteristic line and not just when $t = 0$. hence

$$u(x,t) = 2x_0$$

But $x_0 = x \exp\left(\frac{-t^2}{2}\right)$ from (2), hence

$$u(x,t) = 2x \exp\left(\frac{-t^2}{2}\right)$$

2 Problem 3

Solve

$u_t + (x \sin t)\, u_x = 0$

$u(x, 0) = \frac{1}{1+x^2}$

Solution

Seek solution where $u(s) = u(t(s), x(s)) =$ constant,hence

$$\frac{du}{ds} = \frac{\partial u}{\partial t}\frac{dt}{ds} + \frac{\partial u}{\partial x}\frac{dx}{ds} = 0$$

Compare to (1) we see that $\frac{dt}{ds} = 1$ or $t = s$ and $\frac{dx}{ds} = x \sin t$, but since $t = s$ then $\frac{dx}{ds} = x \sin s$, and this has solution

$$\ln x = \int \sin(s)\, ds$$
$$x = x_0 \exp(-\cos(s))$$

but $s = t$ hence

$$x = x_0 \exp(-\cos(t)) \tag{1}$$

Hence

$$x_0 = x \exp(\cos(t)) \tag{2}$$

At $t = 0$,

$$x = x_0 \exp(-1)$$

Now we are told the solution at $t = 0$ is $\frac{1}{1+x^2}$, or $\frac{1}{1+[x_0 \exp(-1)]^2}$but this solution is valid any where on this characteristic line and not just when $t = 0$. hence

$$u(x,t) = \frac{1}{1 + [x_0 \exp(-1)]^2}$$

Replace the value of x_0 obtained in (2) we obtain

$$\begin{aligned} u(x,t) &= \frac{1}{1 + [x \exp(\cos(t)) \exp(-1)]^2} \\ &= \frac{1}{1 + x^2 \exp(2\cos(t)) \exp(-2)} \end{aligned}$$

Hence

$$u(x,t) = \frac{\exp(2)}{\exp(2) + x^2 \exp(2\cos(t))}$$

3 Problem 4

Solve

$u_t - (tx^2)\, u_x = 0$

$u(x,0) = 1 + x$

Solution

Seek solution where $u(s) = u(t(s), x(s)) =$ constant,hence

$$\frac{du}{ds} = \frac{\partial u}{\partial t}\frac{dt}{ds} + \frac{\partial u}{\partial x}\frac{dx}{ds} = 0$$

Compare to (1) we see that $\frac{dt}{ds} = 1$ or $t = s$ and $\frac{dx}{ds} = -tx^2$, but since $t = s$ then $\frac{dx}{ds} = -sx^2$ hence we need to solve

$$\begin{aligned} \frac{dx}{x^2} &= -sds \\ -\frac{1}{x} &= -\frac{s^2}{2} + x_0 \end{aligned}$$

but $s = t$ hence

$$-\frac{1}{x} = -\frac{t^2}{2} + x_0 \tag{1}$$

Hence

$$x_0 = -\left(\frac{1}{x} - \frac{t^2}{2}\right) \tag{2}$$

At $t = 0$,

$$x_0 = -\frac{1}{x}$$

Now we are told the solution at $t = 0$ is $1 + x$, or $1 - \frac{1}{x_0}$but this solution is valid any where on this characteristic line and not just when $t = 0$. hence

$$u(x,t) = 1 - \frac{1}{x_0}$$

Replace the value of x_0 obtained in (2) we obtain

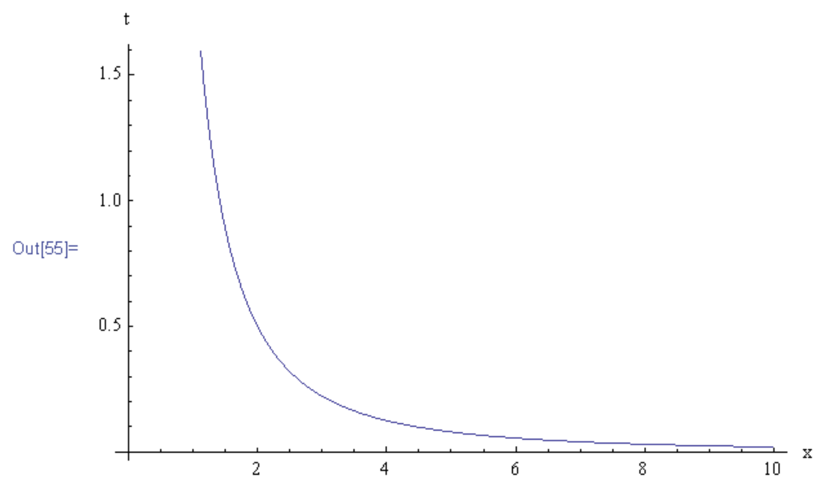
$$\begin{aligned} u(x,t) &= 1 - \frac{1}{-\left(\frac{1}{x} - \frac{t^2}{2}\right)} \\ &= 1 + \frac{2x}{2 - xt^2} \end{aligned}$$

Hence

$$u(x,t) = \frac{2 - xt^2 + 2x}{2 - xt^2}$$

To avoid a solution u which blow up, we need $2 - xt^2 \neq 0$, hence $xt^2 \neq 2$, for example, $x = 2$ and $t = 1$ will not give a valid solution. so all region in $x - t$ plane in which $xt^2 = 2$ is not a valid region to apply this solution at.

The solution breaks down along this line in the $x - t$ plane



To see it in 3D, here is the $u(x,t)$ solution that includes the above line, and we see that the solution below the line and the above the line are not continuous across it. (I think there is a name to this phenomena that I remember reading about sometime, may be related to shockwaves but do not now know how this would happen in reality)

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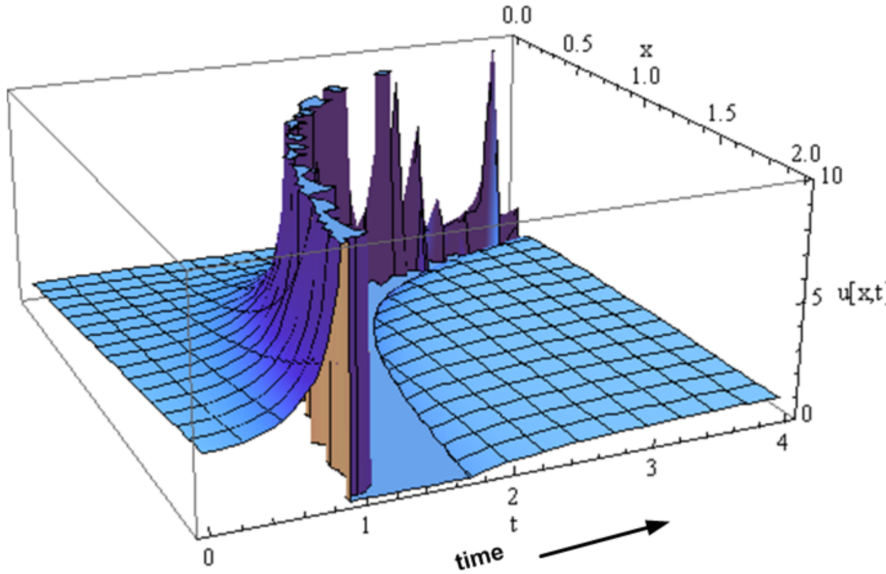

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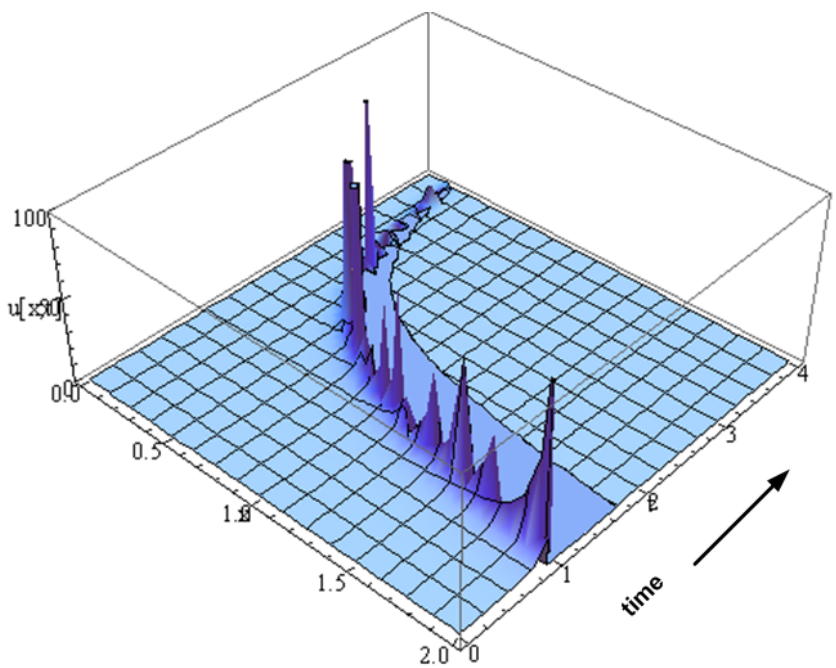

4 Problem 5

$u_t - u_x = xu$

$u(x,0) = 2x$

Solution

Nonhomogeneous pde first order.

(TO DO)