# Math 499, independent studies. California State University, Fullerton. Spring 2007

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# Contents

# Chapter 1

# Introduction

This is a one unit independent studies Math course I took during spring 2007 to help me to practice doing Linear Algebra. Supervisor is

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# Chapter 2

# HWs

## 2.1 HW 1

### 2.1.1 Problem 1

#### Part A

Show that the set of even functions, $f(x) = f(-x)$ is a subspace of the vector space of all function $f(\Re)$

Answer:

(a) if $f$ is an even function, then $f(x) - f(-x) = 0$

let $w(x) = f(x) + g(k)$ where $f, g$ are even functions. To show closure under addition, We need to show that $w(x)$ is also an even function.

$$\begin{aligned} w(x) - w(-x) &= \{f(x) + g(x)\} - \{f(-x) + g(-x)\} \\ &= \{f(x) - f(-x)\} + \{g(x) - g(-x)\} \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Hence $w(x)$ is closed under addition. To show closure under scalar multiplication. Let $c \in \Re$. we need to show that $cf(x)$ is even function when $f(x)$ is even function. Let $g(x) = cf(x)$

$$\begin{aligned} g(x) - g(-x) &= cf(x) - cf(-x) \\ &= c\{f(x) - f(-x)\} \\ &= c(0) \\ &= 0 \end{aligned}$$

Hence closed under scalar multiplication.

And since the "zero" function is also even (and odd as well), Hence even functions are subspace of the vector space of all function $f(\Re)$

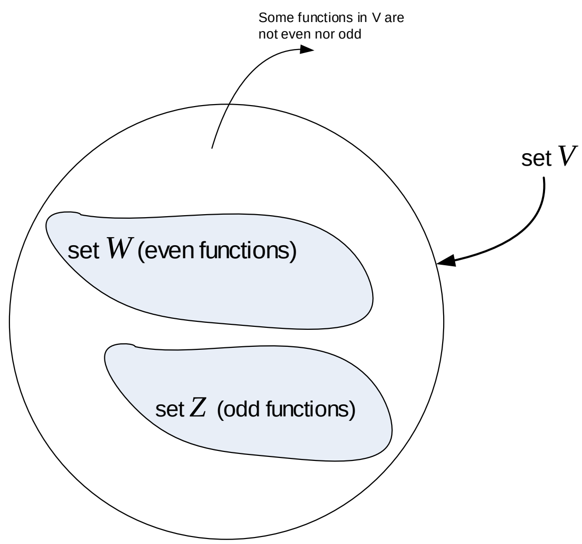
Part B

Show that the set of odd functions, $g(-x) = -g(x)$ form a complementary subspace with the set of even functions (i.e. two subspaces $W, Z$ of $V$ are complementary if

(i) $W \cap Z = \left\{\vec{0}\right\}$

(ii) $W \cup Z = V$, i.e. every $\vec{v} \in V$ can be written as $\vec{v} = \vec{w} + \vec{z}$ where $\vec{w} \in W, \vec{z} \in Z$

**solution:** Let the set of odd functions be $W$ and let the set of even functions be $Z$. Let the set of all functions be $V$.




To show that $W, Z$ are complementary, we need to show that the above 2 properties are met.

Looking at property (i). This property says that the function $\vec{v} \in V$ can be decomposed into the sum of an odd function and even function in one and only one way. i.e. $\vec{v} = \vec{w} + \vec{z}$ where $\vec{w} \in W, \vec{z} \in Z$ is a unique decomposition of $\vec{v}$.

To show this, apply proof by contradiction. Assume the function $\vec{v} \in V$ can be written as the sum of even and odd functions in 2 different ways. $\vec{v} = \vec{w}_1 + \vec{z}_1$ and also $\vec{v} = \vec{w}_2 + \vec{z}_2$ where $\vec{w}_1, \vec{w}_2 \in W$ and $\vec{z}_1, \vec{z}_2 \in Z$. But this means that $\vec{w}_1 + \vec{z}_1 = \vec{w}_2 + \vec{z}_2$. Which implies that $\vec{w}_1 - \vec{w}_2 = \vec{z}_2 - \vec{z}_1$.

Since the difference between 2 even functions is an even function (This can be easily shown from properties of even functions if needed), and the difference between 2 odd function is an odd function, then we have that an even function is identically equal to an odd function. Which is not possible unless both are zero. Hence $\vec{w}_1 - \vec{w}_2 = \vec{z}_2 - \vec{z}_1 = 0$ which means that $\vec{w}_1 = \vec{w}_2$ and $\vec{z}_2 = \vec{z}_1$, therefor the decomposition of $\vec{v}$ must be unique. This proofs property (i).

Now we need to proof property (ii). This means that any function can be written as the sum of an odd and even function.

answer: Let $f(x) \in V$ be any arbitrary function. Write it as follows

$$f(x) = \frac{1}{2}f(x) + \frac{1}{2}f(x)$$

Now add and subtract from the RHS $\frac{1}{2}f(-x)$, This will not change anything

$$f(x) = \frac{1}{2}f(x) + \frac{1}{2}f(x) + \left\{\frac{1}{2}f(-x) - \frac{1}{2}f(-x)\right\}$$

regroup as follows

$$\begin{aligned} f(x) &= \left\{\frac{1}{2}f(x) + \frac{1}{2}f(-x)\right\} + \left\{\frac{1}{2}f(x) - \frac{1}{2}f(-x)\right\} \\ &= \frac{1}{2}\{f(x) + f(-x)\} + \frac{1}{2}\{f(x) - f(-x)\} \end{aligned}$$

Now let $g(x) = \{f(x) + f(-x)\}$, then to show that $g(x)$ is even, i.e. $g(x) \in W$, need to show that $g(x) - g(-x) = 0$

$$\begin{aligned} g(x) - g(-x) &= \{f(x) + f(-x)\} - \{f(-x) + f(-(-x))\} \\ &= \{f(x) + f(-x)\} - \{f(-x) + f(x)\} \\ &= f(x) - f(x) + f(-x) - f(-x) \\ &= 0 \end{aligned}$$

Hence $g(x)$ is even.

Now let $h(x) = \{f(x) - f(-x)\}$, to show that $h(x)$ is odd, i.e. $h(x) \in Z$, we need to show that $h(-x) = -h(x)$ or $h(-x) + h(x) = 0$

$$\begin{aligned} h(-x) + h(x) &= \{f(-x) - f(-(-x))\} + \{f(x) - f(-x)\} \\ &= \{f(-x) - f(x)\} + \{f(x) - f(-x)\} \\ &= f(-x) - f(-x) - f(x) + f(x) \\ &= 0 \end{aligned}$$

Hence $h(x)$ is odd.

Hence we showed that $f(x) = \frac{1}{2}$even function $+\frac{1}{2}$odd function. Hence $f(x) = f_e(x) + f_o(x)$ where $f_e(x)$ is the even part of $f(x)$ and $f_o(x)$ is the odd part of $f(x)$.

side note: Let the basis of the subspace $W$ be $\{w_1, w_2, \cdots, w_n\}$, and let the basis of the subspace $Z$ be $\{z_1, z_2, \cdots, z_n\}$. Property (ii) implies that a basis of $V$ can be taken as the union of these 2 sets of bases, i.e. basis for $V = \{w_1, w_2, \cdots, w_n\} \cup \{z_1, z_2, \cdots, z_n\} = \{w_1, w_2, \cdots, w_n, z_1, z_2, \cdots, z_n\}$

Part C

**Problem:** Show that every function can be uniquely written as the sum of even and odd function.

**Solution:** From part(b), since we showed that the subspaces of even and odd functions are complementary, hence this follows from the property of such subspaces.

### 2.1.2 Problem 2

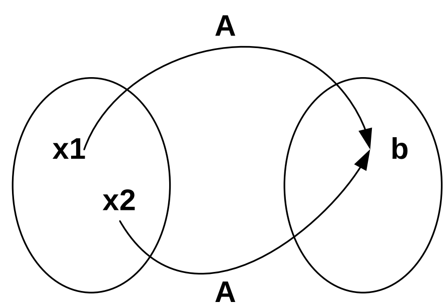
**Problem**: Prove that a linear system $Ax = b$ of $m$ linear equations in $n$ unknowns has either

1. exactly one solution
2. infinitely many solutions
3. no solution

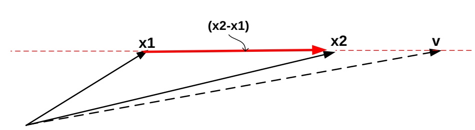
**answer:**

What I have to show is that if more than one solution exist, then there is infinite number of solutions. In other words, one can not have finitely countable number of solutions other than zero or 1.

Assume there exist 2 solutions. $\mathbf{x}_1, \mathbf{x}_2$, hence $A\mathbf{x}_1 = \mathbf{b}$, and $A\mathbf{x}_2 = \mathbf{b}$.




We can show that any point on the line joining the vectors $\mathbf{x}_1, \mathbf{x}_2$ is also a solution.




Vector $\mathbf{v}$ can be parameterized by scalar $t$ where

$$\mathbf{v} = \mathbf{x}_1 + t(\mathbf{x}_2 - \mathbf{x}_1)$$

By changing $t$ we can obtain new vector $\mathbf{v}$. There are infinitely many such vectors as $t$ can have infinitely many values.

$$
\begin{aligned}
A\mathbf{v} &= A(\mathbf{x}_1 + t(\mathbf{x}_2 - \mathbf{x}_1)) \\
&= A(\mathbf{x}_1) + A(t(\mathbf{x}_2 - \mathbf{x}_1)) \quad \text{by linearity of A} \\
&= A(\mathbf{x}_1) + tA(\mathbf{x}_2 - \mathbf{x}_1) \quad \text{by linearity of A} \\
&= A(\mathbf{x}_1) + t(A(\mathbf{x}_2) - A(\mathbf{x}_1)) \quad \text{by linearity of A}
\end{aligned}
$$

But $A(\mathbf{x}_1) = \mathbf{b}$, and $A(\mathbf{x}_2) = \mathbf{b}$, hence the above becomes

$$
\begin{aligned}
A\mathbf{v} &= \mathbf{b} + t(\mathbf{b} - \mathbf{b}) \\
&= \mathbf{b}
\end{aligned}
$$

Therefor, $\mathbf{v}$, which is different than $\mathbf{x}_1$ and $\mathbf{x}_2$ is also a solution. Hence if there are 2 solutions, then we can always find an arbitrary new solution from these 2 solutions, Hence there are infinitely many solutions. QED

### 2.1.3 Problem 3

Problem: Prove that the inner product defined by $\langle f, g\rangle = \int_a^b f(x)g(x) + f'(x)g'(x)dx$ satisfy the conditions of an inner product on the space on continuously differentiable functions on the interval $[a, b]$
Answer:

An inner product must satisfy the following properties. Let $f, g, w$ be continuously differentiable functions on $[a, b]$ and let $t$ be scalar.

1. $\langle f, g\rangle = \langle g, f\rangle$

2. $\langle tf, g\rangle = t\langle f, g\rangle$

3. $\langle f + g, w\rangle = \langle f, w\rangle + \langle g, w\rangle$

4. $\langle f, f\rangle > 0$ if $f \neq 0$ or $\langle f, f\rangle = 0$ iff $f = 0$

To show property 1. Since

$$
\langle f, g\rangle = \int_a^b f(x)g(x) + f'(x)g'(x)dx
$$

Now, since real valued functions are commutative under multiplication (i.e. $f(x)g(x) = g(x)f(x)$) and similarly for the derivatives, we can exchange the order of multiplication

$$
\begin{aligned}
\langle f, g\rangle &= \int_a^b g(x)f(x) + g'(x)f'(x)dx \\
&= \langle g, f\rangle
\end{aligned}
$$

To show property 2:

$$
\begin{aligned}
\langle tf, g\rangle &= \int_a^b tf(x)g(x) + (tf(x))'g'(x)dx \\
&= \int_a^b tf(x)g(x) + tf(x)'g'(x)dx \quad \text{since t is constant} \\
&= \int_a^b t(f(x)g(x) + f(x)'g'(x))dx \\
&= t\int_a^b f(x)g(x) + f(x)'g'(x)dx \\
&= t\,\langle f, g\rangle
\end{aligned}
$$

To show property 3:

$$
\begin{aligned}
\langle f+g, w\rangle &= \int_a^b (f+g)(x)\ w(x) + \frac{d}{dx}(f+g)(x)\ w'(x)dx \\
&= \int_a^b (f(x)+g(x))\ w(x) + (f'(x)+g'(x))\ w'(x)dx
\end{aligned}
$$

Now, since we can distribute multiplication over addition for real valued functions, i.e. $(f+g)w = fw + gw$ (because function multiplications is a point-by-point multiplication) the above becomes

$$
\langle f+g, w\rangle = \int_a^b \{f(x)\ w(x) + g(x)\ w(x)\} + \{f'(x)w'(x) + g'(x)w'(x)\}\ dx
$$

By linearity of integration operation we can break above integral into the sum of two integrals

$$
\begin{aligned}
\langle f+g, w\rangle &= \int_a^b f(x)\ w(x) + f'(x)w'(x)dx + \int_a^b g(x)\ w(x) + g'(x)w'(x)\ dx \\
&= \langle f, w\rangle + \langle g, w\rangle
\end{aligned}
$$

To show property 4:

$$
\begin{aligned}
\langle f, f\rangle &= \int_a^b f(x)\ f(x) + f'(x)f'(x)dx \\
&= \int_a^b [f(x)]^2 + [f'(x)]^2\ dx \\
&= \int_a^b [f(x)]^2\ dx + \int_a^b [f'(x)]^2\ dx
\end{aligned}
$$

Consider $\int_a^b [f(x)]^2\ dx$. Since $[f(x)]^2$ can only be positive or zero,This is the same as $\int_a^b g(x)dx$ where $g(x) \geq 0$ over $[a, b]$, Hence $\int_a^b g(x)dx = 0$ only if $g(x)$ is identically zero over $[a, b]$, but if $g(x) = 0$, then $[f(x)]^2 = 0$ or $f(x) = 0$, which means $\int_a^b [f(x)]^2\ dx = 0$.

Now if $f(x) = 0$, then the second integral $\int_a^b [f'(x)]^2\, dx = 0$ as well.

Hence $\langle f, f\rangle = 0$ only if $f(x)$ is identically zero over $[a, b]$

Hence we showed the 4 properties for this definition of the inner product.

### 2.1.4 Problem 4

**problem:** $L_2$ norm on the interval $[a, b]$ is defined as $\langle f, f\rangle = \int_b^a [f(x)]^2\, dx$

Find the cubic polynomial that best approximates the function $e^x$ on the interval $[0, 1]$ by minimizing the $L_2$ error.

**solution:**

Let $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$, hence we need to have 4 equations to solve for $a_0, a_1, a_2, a_3$

Let the $g(x) = p(x) - e^x$, which is the error function.

From the definition, the square of norm of this error is

$$\begin{aligned}
|E|^2 &= \|p(x) - e^x\|^2 \\
&= \|g(x)\|^2 \\
&= \langle g(x), g(x)\rangle \\
&= \int_0^1 [g(x)]^2\, dx \\
&= \int_0^1 [p(x) - e^x]^2\, dx
\end{aligned}$$

$$\begin{aligned}
|E|^2 = \int_0^1 [p(x) - e^x]^2\, dx &= \int_0^1 \left[a_0 + a_1x + a_2x^2 + a_3x^3 - e^x\right]^2 dx \\
&= -\frac{1}{2} + \frac{e^2}{2} + 2a_0 + a_0^2 + a_0a_1 + \frac{a_1^2}{3} + 4a_2 + \\
&\frac{2a_0a_2}{3} + \frac{a_2^2}{5} + a_1(-2 + \frac{a_2}{2} + \frac{2a_3}{5}) - 12a_3 + \\
&\frac{a_0a_3}{2} + \frac{a_2a_3}{3} + \frac{a_3^2}{7} + e(-2a_0 - 2a_2 + 4a_3)
\end{aligned}$$

Now minimize this error with respect to each of the coefficients in turn to generate 4 equations to solve.

$$\frac{d|E|^2}{da_0} = 0 = 2 - 2e + 2a_0 + a_1 + \frac{2a_2}{3} + \frac{a_3}{2}$$
$$\frac{d|E|^2}{da_1} = 0 = -2 + a_0 + \frac{2a_1}{3} + \frac{a_2}{2} + \frac{2a_3}{5}$$
$$\frac{d|E|^2}{da_2} = 0 = 4 - 2e + \frac{2a_0}{3} + \frac{a_1}{2} + \frac{2a_2}{5} + \frac{a_3}{3}$$
$$\frac{d|E|^2}{da_4} = 0 = -12 + 4e + \frac{a_0}{2} + \frac{2a_1}{5} + \frac{a_2}{3} + \frac{2a_3}{7}$$

Hence, set up the above 4 equations in matrix form, we obtain

$$\begin{bmatrix} 2 & 1 & \frac{2}{3} & \frac{1}{2} \\ 1 & \frac{2}{3} & \frac{1}{2} & \frac{2}{5} \\ \frac{2}{3} & \frac{1}{2} & \frac{2}{5} & \frac{1}{3} \\ \frac{1}{2} & \frac{2}{5} & \frac{1}{3} & \frac{2}{7} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 2e - 2 \\ 2 \\ 2e - 4 \\ 12 - 4e \end{bmatrix}$$

Solving for $a's$ using Gaussian elimination leads to solution

$$a_0 = 0.9906$$
$$a_1 = 1.0183$$
$$a_2 = 0.421246$$
$$a_3 = 0.278625$$

Hence the best fit cubic polynomial that minimize the error to $e^x$ between 0 and 1 is

$$p(x) = 0.9906 + 1.0183x + 0.421246x^2 + 0.278625x^3$$

This is a table of values to compare $e^x$ and $p(x)$

```
In[29]:= Table[{x, Exp[x], p[sol, x]}, {x, 0, 1, .1}];
         TableForm[%, TableHeadings -> {None, {"x", "p(x)", "exp(x)"}}]
Out[30]//TableForm=
```

| x | p(x) | exp(x) |
|---|---|---|
| 0 | 1 | 0.99906 |
| 0.1 | 1.10517 | 1.10538 |
| 0.2 | 1.2214 | 1.2218 |
| 0.3 | 1.34986 | 1.34999 |
| 0.4 | 1.49182 | 1.49161 |
| 0.5 | 1.64872 | 1.64835 |
| 0.6 | 1.82212 | 1.82187 |
| 0.7 | 2.01375 | 2.01385 |
| 0.8 | 2.22554 | 2.22595 |
| 0.9 | 2.4596 | 2.45986 |
| 1. | 2.71828 | 2.71723 |

### 2.1.5 Problem 5

**Problem:** A Hilbert space is a function space with a norm. If we consider the space of continuous functions on $[a, b]$ with $L_2$ norm, it is Hilbert space $H$. A key step in showing that functions on this space can be approximated using a countable (i.e. indexed by integers) orthonormal set is the Bessel Inequality

$$\sum_{i=1}^{n} \langle f, \phi_i \rangle^2 \leq \|f\|^2 < \infty$$

where $\phi_i$ is an element of the orthonormal set and $f$ is the element of the Hilbert space being approximated.

If we approximate $f(x)$ by $\sum_{i=1}^{n} \alpha_i \phi_i(x)$ with $\alpha_i = \langle f, \phi_i \rangle$. Start by stating the error in the approximation to prove the Bessel inequality.

**solution**

In this solution, I use the analogy to the normal Euclidean space just as a guideline.

$\alpha_i = \langle f, \phi_i \rangle$ is the projection of the function $f$ onto the basis $\phi_i$. This is similar to extracting the $i^{th}$ coordinate of a vector. The expression $\alpha_i \phi_i(x)$ is then a vector along the direction of the base $\phi_i$, whose length is the projection of $f$ in the direction of the $i^{th}$ basis. Hence in general,

$$f(x) = \sum_{i=1}^{Number\ of\ Basis} \alpha_i \phi_i(x)$$

This is similar to the Euclidean coordinate system where we write $\vec{v} = x\vec{i} + y\vec{j} + z\vec{k}$ where $\vec{i}, \vec{j}, \vec{k}$ are the basis in this space and $x, y, z$ are the coordinates of the vector. A vector coordinate is the length of the projection of the vector onto each specific basis. The expression for $f(x)$ above is a generalization of this concept to the function space and to an arbitrary number of basis.

And similarly to what we do in the Euclidean space, the 'length' of the vector using $L_2$ norm is $\|\vec{v}\|_2 = \sqrt{x^2 + y^2 + z^2}$, hence $\|\vec{v}\|_2^2 = x^2 + y^2 + z^2$. This is generalized to the $H$ space by saying

$$\begin{aligned}
\|f\|^2 &= \sum_{i}^{Number\ of\ Basis} (\alpha_i)^2 \\
&= \sum_{i}^{Number\ of\ Basis} \langle f, \phi_i \rangle^2
\end{aligned}$$

If the number of basis is infinite, then we write

$$\|f\|^2 = \sum_{i}^{\infty} \langle f, \phi_i \rangle^2$$

Therefore, if the number of basis is infinite, and we sum for some finite number of basis less than infinite, say $n$, hence the resulting norm must be less than the actual norm we would get if we had added over all the basis. Hence it is obvious that $\|f\|^2 \geq \sum_{i}^{n} \langle f, \phi_i \rangle^2$ since we terminated the sum earlier, and since each quantity being summed is positive, then the partial sum must be less than the limit, which is $\|f\|^2$.

Now we just need to show that the norm finite. If the function itself is finite (meaning its value, or range, is finite) then each of its projections must be finite ($|\cos\alpha| \leq 1$). Hence given a function which does not "blow" up, then all its components must be finite. Since we are adding finite number of quantities, each of which is finite in its own, hence the sum must be finite as well. Hence $\|f\| < \infty$, or $\|f\|^2 < \infty$

Therefore

$$\sum_i^n \langle f, \phi_i \rangle^2 \leq \|f\|^2 < \infty$$

## 2.2 HW 2

### 2.2.1 Problem

question: Consider the solution of $A\mathbf{x} = \mathbf{b} + \mathbf{n}$ where $A$ is $m \times n$ matrix, $\mathbf{b}\epsilon R^m$, $\mathbf{x}\epsilon R^n$ and $\mathbf{n}$ is vector of i.i.d. Gaussian $N\left(0, \sigma^2 \mathbf{I}_m\right)$ noise vector. (i.e. noise is white Gaussian noise). Determine the best solution $\vec{x}$

**Answer**

Let us refer to the observed output (which includes the noise $\mathbf{n}$) as $\mathbf{z}$, hence we write $\mathbf{z} = \mathbf{b} + \mathbf{n}$ where $\mathbf{b}$ is the uncontaminated output (what the observed output would be if there is no noise).

Since the noise $\mathbf{n}$ is an additive noise to the output $\mathbf{b}$ of the system, the since the noise has zero mean, then the mean of $\mathbf{z}$ will be the same as the mean of $\mathbf{b}$. But $\mathbf{b}$ is a deterministic signal which does not change, hence its mean is its value, hence the mean of $\mathbf{z}$ is $\mathbf{b}$.

Now, $\mathbf{z}$ is described by a probability density function PDF as follows ( $\mathbf{z}$ is in $R^m$, hence it is $m$ long vector)

$$\begin{aligned}
\Pr\left(\mathbf{z}; \left\{\mu(\mathbf{z}), \sigma^2 \mathbf{I}_m\right\}\right) &= \frac{1}{\left(2\pi\sigma^2\right)^{\frac{m}{2}}} \exp\left(-\frac{1}{2\sigma^2} \|\mathbf{z} - \mu(\mathbf{z})\|^2\right) \\
\Pr\left(\mathbf{z}; \left\{\mathbf{b}, \sigma^2 \mathbf{I}_m\right\}\right) &= \frac{1}{\left(2\pi\sigma^2\right)^{\frac{m}{2}}} \exp\left(-\frac{1}{2\sigma^2} \|\mathbf{z} - \mathbf{b}\|^2\right)
\end{aligned}$$

But since $A\mathbf{x} = \mathbf{b}$, then the above can be written as

$$\Pr\left(\mathbf{z}; \left\{A\mathbf{x}, \sigma^2 \mathbf{I}_m\right\}\right) = \frac{1}{\left(2\pi\sigma^2\right)^{\frac{m}{2}}} \exp\left(-\frac{1}{2\sigma^2} \|\mathbf{z} - A\mathbf{x}\|^2\right)$$

Since $A$ is a constant matrix (system is assumed to be time-invariant), hence from the above we see that the expression gives the probability of observing $\mathbf{z}$ for a given $\mathbf{x}$. Hence the best estimate of $\mathbf{x}$ would be the one which maximizes this probability. Instead of maximizing the PDF directly, we maximize its natural logarithm (a mathematical convenience trick, no more).

Now find the natural logarithm of the above quantity, and find where the result is maximum.

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{x}} \ln \Pr\left(\mathbf{z}; \left\{A\mathbf{x}, \sigma^2 \mathbf{I}_m\right\}\right) &= \frac{\partial}{\partial \mathbf{x}} \left(\ln \frac{1}{\left(2\pi\sigma^2\right)^{\frac{m}{2}}} \exp\left(-\frac{1}{2\sigma^2} \left\|\mathbf{z} - A\mathbf{x}\right\|^2\right)\right) \\
&= \nabla_{\mathbf{x}} \left(\ln \frac{1}{\left(2\pi\sigma^2\right)^{\frac{m}{2}}} + \ln \exp\left(-\frac{1}{2\sigma^2} \left\|\mathbf{z} - A\mathbf{x}\right\|^2\right)\right) \\
&= \overbrace{\nabla_{\mathbf{x}} \left(\ln \frac{1}{\left(2\pi\sigma^2\right)^{\frac{m}{2}}}\right)}^{0} + \nabla_{\mathbf{x}} \left(-\frac{1}{2\sigma^2} \left\|\mathbf{z} - A\mathbf{x}\right\|^2\right) \\
&= -\frac{1}{2\sigma^2} \nabla_{\mathbf{x}} \left(\left\|\mathbf{z} - A\mathbf{x}\right\|^2\right) \qquad (1)
\end{aligned}$$

Start with $m = n = 1$, hence the above becomes[1]

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{x}} \ln \Pr\left(\mathbf{z}; \left\{A\mathbf{x}, \sigma^2 \mathbf{I}_m\right\}\right) &= -\frac{1}{2\sigma^2} \nabla_{x_1} \left(z_1^2 + a_{11}^2 x_1^2 - 2 z_1 a_{11} x_1\right) \\
&= -\frac{1}{2\sigma^2} \left(2 a_{11}^2 x_1 - 2 a_{11} z_1\right) \\
&= -\frac{1}{\sigma^2} \left(a_{11}^2 x_1 - a_{11} z_1\right)
\end{aligned}$$

Set the above to zero and solve for $x_1$

$$\begin{aligned}
0 &= \left(a_{11} x_1 - z_1\right) \\
x_1 &= \frac{z_1}{a_{11}}
\end{aligned}$$

Hence

$$\boxed{x_1 = \frac{z_1}{a_1 1}}$$

This matches the least squares solution $x_1 = \left(A^T A\right)^{-1} A^T z_1 \rightarrow x_1 = \left(a_{11} a_{11}\right)^{-1} a_{11} z_1 = \frac{z_1}{a_{11}}$

Now I need to do this for $m = n = 2$, and assuming that $var\left(z_1\right) = var\left(z_1\right)$. We can start from equation (1) above, shown again below

[1] I will start the index at 1 to be Matlab friendly

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{x}} \ln \Pr\left(\mathbf{z}; \left\{A\mathbf{x}, \sigma^2 \mathbf{I}_m\right\}\right) &= -\frac{1}{2\sigma^2} \nabla_{\mathbf{x}} \left(\|\mathbf{z} - A\mathbf{x}\|^2\right) \\
-2\sigma^2 \frac{\partial}{\partial \mathbf{x}} \ln \Pr\left(\mathbf{z}; \left\{A\mathbf{x}, \sigma^2 \mathbf{I}_m\right\}\right) &= \nabla_{\mathbf{x}} \left(\|\mathbf{z} - A\mathbf{x}\|^2\right) \\
&= \nabla_{\mathbf{x}} \left( \left\| \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|_2^2 \right) \\
&= \nabla_{\mathbf{x}} \left( \left\| \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} - \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix} \right\|_2^2 \right) \\
&= \nabla_{\mathbf{x}} \left( \left\| \begin{pmatrix} z_1 - (a_{11}x_1 + a_{12}x_2) \\ z_2 - (a_{21}x_1 + a_{22}x_2) \end{pmatrix} \right\|_2^2 \right) \\
&= \nabla_{\mathbf{x}} \left( [z_1 - (a_{11}x_1 + a_{12}x_2)]^2 + [z_2 - (a_{21}x_1 + a_{22}x_2)]^2 \right)
\end{aligned}$$

Let

$$\begin{aligned}
f(x_1, x_2) &= [z_1 - (a_{11}x_1 + a_{12}x_2)]^2 + [z_2 - (a_{21}x_1 + a_{22}x_2)]^2 \\
&= \left[z_1^2 + (a_{11}x_1 + a_{12}x_2)^2 - 2z_1(a_{11}x_1 + a_{12}x_2)\right] + \left[z_2^2 + (a_{21}x_1 + a_{22}x_2)^2 - 2z_2(a_{21}x_1 + a_{22}x_2)\right] \\
&= \left[z_1^2 + a_{11}^2x_1^2 + a_{12}^2x_2^2 + 2a_{11}a_{12}x_1x_2 - 2z_1a_{11}x_1 - 2z_1a_{12}x_2\right] + \\
&\left[z_2^2 + a_{21}^2x_1^2 + a_{22}^2x_2^2 + 2a_{21}a_{22}x_1x_2 - 2z_2a_{21}x_1 - 2z_2a_{22}x_2\right]
\end{aligned}$$

Then

$$\nabla_{\mathbf{x}} (f(x_1, x_2)) = \begin{pmatrix} \frac{\partial f(x_1, x_2)}{\partial x_1} \\ \frac{\partial f(x_1, x_2)}{\partial x_2} \end{pmatrix}$$

so

$$\begin{aligned}
\frac{\partial f(x_1, x_2)}{\partial x_1} &= \left[0 + 2a_{11}^2x_1 + 0 + 2a_{11}a_{12}x_2 - 2z_1a_{11} - 0\right] + \left[0 + 2a_{21}^2x_1 + 0 + 2a_{21}a_{22}x_2 - 2z_2a_{21} - 0\right] \\
&= 2a_{11}^2x_1 + 2a_{11}a_{12}x_2 - 2z_1a_{11} + 2a_{21}^2x_1 + 2a_{21}a_{22}x_2 - 2z_2a_{21} \\
&= x_1\left(2a_{11}^2 + 2a_{21}^2\right) + x_2(2a_{11}a_{12} + 2a_{21}a_{22}) - 2z_1a_{11} - 2z_2a_{21}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial f(x_1, x_2)}{\partial x_2} &= \left[0 + 0 + 2a_{12}^2x_2 + 2a_{11}a_{12}x_1 - 0 - 2z_1a_{12}\right] + \left[0 + 0 + 2a_{22}^2x_2 + 2a_{21}a_{22}x_1 - 0 - 2z_2a_{22}\right] \\
&= 2a_{12}^2x_2 + 2a_{11}a_{12}x_1 - 2z_1a_{12} + 2a_{22}^2x_2 + 2a_{21}a_{22}x_1 - 2z_2a_{22} \\
&= x_1(2a_{11}a_{12} + 2a_{21}a_{22}) + x_2\left(2a_{12}^2 + 2a_{22}^2\right) - 2z_1a_{12} - 2z_2a_{22}
\end{aligned}$$

Hence we obtain that

$$\frac{\partial}{\partial \mathbf{x}} \ln \Pr\left(\mathbf{z}; \left\{A\mathbf{x}, \sigma^2 \mathbf{I}_m\right\}\right) = -\frac{1}{2\sigma^2} \nabla_{\mathbf{x}} \left(f\left(x_1, x_2\right)\right)$$
$$= -\frac{1}{2\sigma^2} \begin{pmatrix} x_1\left(2a_{11}^2 + 2a_{21}^2\right) + x_2\left(2a_{11}a_{12} + 2a_{21}a_{22}\right) - 2z_1a_{11} - 2z_2a_{21} \\ x_1\left(2a_{11}a_{12} + 2a_{21}a_{22}\right) + x_2\left(2a_{12}^2 + 2a_{22}^2\right) - 2z_1a_{12} - 2z_2a_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence

$$x_1\left(2a_{11}^2 + 2a_{21}^2\right) + x_2\left(2a_{11}a_{12} + 2a_{21}a_{22}\right) - 2z_1a_{11} - 2z_2a_{21} = 0$$

and

$$x_1\left(2a_{11}a_{12} + 2a_{21}a_{22}\right) + x_2\left(2a_{12}^2 + 2a_{22}^2\right) - 2z_1a_{12} - 2z_2a_{22} = 0$$

so, solve for $x_1, x_2$. Write as $Ax = b$ and solve:

$$\begin{pmatrix} \left(2a_{11}^2 + 2a_{21}^2\right) & \left(2a_{11}a_{12} + 2a_{21}a_{22}\right) \\ \left(2a_{11}a_{12} + 2a_{21}a_{22}\right) & \left(2a_{12}^2 + 2a_{22}^2\right) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2z_1a_{11} + 2z_2a_{21} \\ 2z_1a_{12} + 2z_2a_{22} \end{pmatrix}$$

Solve for

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \left(2a_{11}^2 + 2a_{21}^2\right) & \left(2a_{11}a_{12} + 2a_{21}a_{22}\right) \\ \left(2a_{11}a_{12} + 2a_{21}a_{22}\right) & \left(2a_{12}^2 + 2a_{22}^2\right) \end{pmatrix}^{-1} \begin{pmatrix} 2z_1a_{11} + 2z_2a_{21} \\ 2z_1a_{12} + 2z_2a_{22} \end{pmatrix}$$
$$= \frac{\begin{pmatrix} \left(2a_{12}^2 + 2a_{22}^2\right) & -\left(2a_{11}a_{12} + 2a_{21}a_{22}\right) \\ -\left(2a_{11}a_{12} + 2a_{21}a_{22}\right) & \left(2a_{11}^2 + 2a_{21}^2\right) \end{pmatrix} \begin{pmatrix} 2z_1a_{11} + 2z_2a_{21} \\ 2z_1a_{12} + 2z_2a_{22} \end{pmatrix}}{\left(2a_{11}^2 + 2a_{21}^2\right)\left(2a_{12}^2 + 2a_{22}^2\right) - \left(2a_{11}a_{12} + 2a_{21}a_{22}\right)\left(2a_{11}a_{12} + 2a_{21}a_{22}\right)}$$

$$= \frac{\begin{pmatrix} \left(2a_{12}^2 + 2a_{22}^2\right)\left(2z_1a_{11} + 2z_2a_{21}\right) - \left(2a_{11}a_{12} + 2a_{21}a_{22}\right)\left(2z_1a_{12} + 2z_2a_{22}\right) \\ -\left(2a_{11}a_{12} + 2a_{21}a_{22}\right)\left(2z_1a_{11} + 2z_2a_{21}\right) + \left(2a_{11}^2 + 2a_{21}^2\right)\left(2z_1a_{12} + 2z_2a_{22}\right) \end{pmatrix}}{\left(2a_{11}^2 + 2a_{21}^2\right)\left(2a_{12}^2 + 2a_{22}^2\right) - \left(2a_{11}a_{12} + 2a_{21}a_{22}\right)\left(2a_{11}a_{12} + 2a_{21}a_{22}\right)}$$

$$= \frac{\begin{pmatrix} 4a_{11}z_1a_{22}^2 + 4a_{12}^2a_{21}z_2 - 4a_{11}a_{12}z_2a_{22} - 4z_1a_{12}a_{21}a_{22} \\ 4z_1a_{12}a_{21}^2 + 4a_{11}^2z_2a_{22} - 4a_{11}z_1a_{21}a_{22} - 4a_{11}a_{12}a_{21}z_2 \end{pmatrix}}{4a_{11}^2a_{22}^2 - 8a_{11}a_{12}a_{21}a_{22} + 4a_{12}^2a_{21}^2}$$

$$= \begin{pmatrix} \frac{4a_{11}z_1a_{22}^2 + 4a_{12}^2a_{21}z_2 - 4a_{11}a_{12}z_2a_{22} - 4z_1a_{12}a_{21}a_{22}}{4a_{11}^2a_{22}^2 - 8a_{11}a_{12}a_{21}a_{22} + 4a_{12}^2a_{21}^2} \\ \frac{4z_1a_{12}a_{21}^2 + 4a_{11}^2z_2a_{22} - 4a_{11}z_1a_{21}a_{22} - 4a_{11}a_{12}a_{21}z_2}{4a_{11}^2a_{22}^2 - 8a_{11}a_{12}a_{21}a_{22} + 4a_{12}^2a_{21}^2} \end{pmatrix}$$

so

$$x_1 = \frac{4a_{11}z_1a_{22}^2 + 4a_{12}^2a_{21}z_2 - 4a_{11}a_{12}z_2a_{22} - 4z_1a_{12}a_{21}a_{22}}{4a_{11}^2a_{22}^2 - 8a_{11}a_{12}a_{21}a_{22} + 4a_{12}^2a_{21}^2}$$
$$= \left(\frac{z_1a_{22} - a_{12}z_2}{a_{11}a_{22} - a_{12}a_{21}}\right)$$

and

$$x_2 = \frac{4z_1a_{12}a_{21}^2 + 4a_{11}^2z_2a_{22} - 4a_{11}z_1a_{21}a_{22} - 4a_{11}a_{12}a_{21}z_2}{4a_{11}^2a_{22}^2 - 8a_{11}a_{12}a_{21}a_{22} + 4a_{12}^2a_{21}^2}$$
$$= \left(\frac{a_{11}z_2 - z_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}}\right)$$

Hence

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \left(\frac{z_1a_{22}-a_{12}z_2}{a_{11}a_{22}-a_{12}a_{21}}\right) \\ \left(\frac{a_{11}z_2-z_1a_{21}}{a_{11}a_{22}-a_{12}a_{21}}\right) \end{pmatrix} \tag{2}$$

is the least squares error. To validate

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^T \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right)^{-1} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^T \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$
$$= \left(\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right)^{-1} \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$
$$= \begin{pmatrix} a_{11}^2 + a_{21}^2 & a_{11}a_{12} + a_{21}a_{22} \\ a_{11}a_{12} + a_{21}a_{22} & a_{12}^2 + a_{22}^2 \end{pmatrix}^{-1} \begin{pmatrix} a_{11}z_1 + a_{21}z_2 \\ z_1a_{12} + z_2a_{22} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{(a_{12}^2+a_{22}^2)}{(a_{11}a_{22}-a_{12}a_{21})^2} & -\frac{a_{11}a_{12}+a_{21}a_{22}}{(a_{11}a_{22}-a_{12}a_{21})^2} \\ -\frac{a_{11}a_{12}+a_{21}a_{22}}{(a_{11}a_{22}-a_{12}a_{21})^2} & \frac{(a_{11}^2+a_{21}^2)}{(a_{11}a_{22}-a_{12}a_{21})^2} \end{pmatrix} \begin{pmatrix} a_{11}z_1 + a_{21}z_2 \\ z_1a_{12} + z_2a_{22} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{(z_1a_{22}-a_{12}z_2)}{a_{11}a_{22}-a_{12}a_{21}} \\ \frac{a_{11}z_2-z_1a_{21}}{a_{11}a_{22}-a_{12}a_{21}} \end{pmatrix} \tag{3}$$

Compare equations (2) and (3) above, they are the same. OK, confirmed.

### 2.2.2 appendix

Some definitions:

$\sigma$ : standard deviation of a distribution or random variable.

Var : Variance. The square of $\sigma$ or $\sigma^2$, the larger the variance, larger the spread about it.

$E(x_i)$: Expected value of random variable $= \mu(x_i)$, its mean.

$$= \sum_{\text{all } x} x\, P(x) \quad \leftarrow \text{probability of } x$$

Covariance: measures how much one random variable varies together with another random variable.

$$cov(x,y) = E[\underbrace{(x - E(x))}_{\text{error in } x}\ \underbrace{(y - E(y))}_{\text{error in } y}]$$

$$= E[\,(x - \mu(x))\ (y - \mu(y))\,]$$

$$= E(xy) - E(x)E(y).$$

note, $cov(x,y) = 0$ if $x, y$ are Linearly independent random variables

(1b) Weighted Least squares normal equations

$$(A^T W W^T A)\ \hat{X}_w = A^T W W^T b$$

where $WW^T = Cov^{-1} = C$

So Linearly independent observation vector $b$, $CoV(b_i, b_j)$

$$= \begin{bmatrix} E[(b_1 - E(b_1))(b_1 - E(b_1))] & 0 & 0 \cdots 0 \\ 0 & E[(b_2 - E(b_2))(b_2 - E(b_2))] & \\ \vdots & & \ddots \\ 0 & & E(b_m - E(b_m))(\ ) \end{bmatrix}$$

Since mean is given as zero (i.e. $E(b_i) = 0$). Then

$$Cov(b) = \begin{bmatrix} E(b_1^2) & 0 & 0 \cdots 0 \\ \vdots & E(b_2^2) & & \\ & & E(b_3^2) & \\ 0 & & & E(b_m^2) \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \ddots \\ & & & \sigma_m^2 \end{bmatrix}$$

Expected value of square of the observation is its variance since cov(b) is diagonal, then $C = Cov^{-1} = \begin{bmatrix} \frac{1}{\sigma_1^2} & & & \\ & \frac{1}{\sigma_2^2} & & \\ & & \ddots & \\ & & & \frac{1}{\sigma_m^2} \end{bmatrix}$

hence $\boxed{\hat{x}_w = (A^T C A)^{-1} A^T C\, b}$

Solution when noise is present

Since A is known, C can be found since $\sigma^2$ is given for the noise, and b is known (this is the observation itself) then we can calculate the above to find $\hat{X}_w$. #

## 2.3 HW 3

### 2.3.1 Problem 1

Question: By setting the derivative to zero, find the value of $b_1$ that minimizes

$$\|b_1 \sin x - \cos x\|^2 = \int_0^{2\pi} (b_1 \sin x - \cos x)^2 \, dx$$

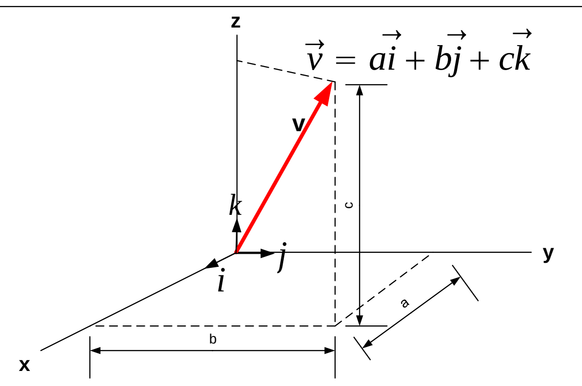
compare with the Fourier coefficient $b_1$

**Answer:**

First I thought it might be a good idea to refresh myself with Fourier series and how it comes about from geometrical perspective. Understanding how a function can be represented using Fourier series can be made easier by making an analogy of how a vector is represented using vector basis.

We know from basic Euclidean geometry, that a vector in the standard 3 dimensional space is written as the sum of its projections on the 3 basis vectors. When we write $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$, then in this case, $a$ is the projection of the vector $\vec{v}$ onto the direction of the base vector $\vec{i}$, and similarly for the numbers $b$ and $c$. The numbers $a, b, c$ are called the *coordinates* of the vector $\vec{v}$ in this particular coordinate system.

The same vector $\vec{v}$ can then have different coordinate values depending on which coordinates system we are making our measurements in, but it the same exact vector. Hence a vector is invariant under coordinate transformation, but its representation (the coordinates) will change. This diagram below illustrates the above




Representing a vector as the sum of its projections on the coordinates system

Now that we know how a vector is represented by adding its projections along the direction of each base vector, we are ready to make the switch to a new and exciting world, where vectors become functions and the number of basis instead of being fixed at 3 become very large, in fact, it become infinitely large. This new vector space is called the Hilbert space.

Our goal is to express, or represent a function such as $f(x)$ using as basis the functions sin and cos. This leads to Fourier series representation of a function. One of the issues to consider right away, is what basis to use. There are many families of basis to select. Here we select the *sin* and *cos* functions as the basis.

As long as each base is orthogonal to each other (using a new definition of what it means to have two functions orthogonal to each others).

Hence by selecting $\sin(x), \sin(2x), \sin(3x), \cdots$, and $\cos(x), \cos(2x), \cdots$. I.e. $\sin(nx), \cos(nx)$ for $n$ over all the integers from $0 \cdots \infty$. These basis work since any two different basis have zero as their dot product using the following definition of dot product, therefore they are orthogonal to each others.

In Hilbert space, two functions are orthogonal to each others if their dot product is zero, defined as follows between the function $f(x), g(x)$

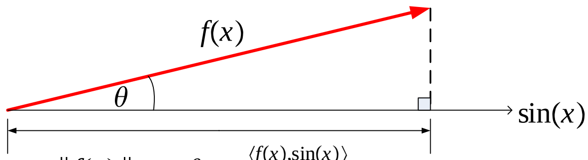
$$\langle f(x), g(x) \rangle = \int_0^{2\pi} f(x) g(x) \, dx$$

So, now when we are given a function $f(x)$ and asked for its representation with respect to the coordinate system called the fourier coordinates system, we follow the same idea as with normal vectors, and write

$$\begin{aligned} f(x) = &\ (\text{projection of } f(x) \text{ onto first basis}) \times \text{first basis} \\ &+ (\text{projection of } f(x) \text{ onto second basis}) \times \text{second basis} \\ &+ \cdots \end{aligned}$$

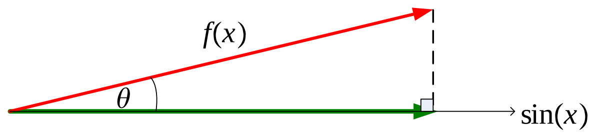
The above is the same as we did with Euclidean space. We now need to know how to find a projection of a function such as $f(x)$ onto a base function such as $\sin(x)$. This diagram shows how to do find one such projection of $f(x)$ onto one base function $\sin(x)$

Step1: Find the length of the projection of a function f(x) on one of the basis (sin(x) in this example).




$$\|f(x)\| \cos\theta = \frac{\langle f(x), \sin(x)\rangle}{\|\sin(x)\|}$$

Step2: To turn the projection (which is just a number) into a vector, we need to multiply this number by a unit vector along the same direction. This unit vector is found by dividing the "vector" sin(x) by the norm of the "vector" sin(x). This gives us the vector "p", which is the projection vector of f(x) onto sin(x)




$$\vec{p} = \frac{\langle f(x), \sin(x)\rangle}{\langle \sin(x), \sin(x)\rangle} \sin(x)$$

The derivation of the projection vector P is shown below

$$\vec{p} = \overbrace{\frac{\langle f(x), \sin(x)\rangle}{\|\sin(x)\|}}^{\text{Length of projection}} \overbrace{\left(\frac{\sin(x)}{\|\sin(x)\|}\right)}^{\text{unit vector along sin(x)}}$$

$$\vec{p} = \frac{\langle f(x), \sin(x)\rangle}{\|\sin(x)\|^2} \sin(x)$$

$$\vec{p} = \frac{\langle f(x), \sin(x)\rangle}{\langle \sin(x), \sin(x)\rangle} \sin(x)$$

The above tells us that the *coordinate* of $f(x)$ along $\sin(x)$ is given by

$$\frac{\langle f(x), \sin(x)\rangle}{\langle \sin(x), \sin(x)\rangle}$$

Let us express $f(x)$ using the first few coordinates. The first base is $\cos(0x) = 1$, the second base is $\cos(x)$, the third base is $\cos(2x)$, etc... and now for the sin basis, again we use $\sin(x), \sin(2x), \cdots$. Hence we have

$$f(x) = \frac{\langle f(x), \cos(0x)\rangle}{\langle \cos(0x), \cos(0x)\rangle} \cos(0x) + \frac{\langle f(x), \cos(x)\rangle}{\langle \cos(x), \cos(x)\rangle} \cos(x) + \frac{\langle f(x), \cos(2x)\rangle}{\langle \cos(2x), \cos(2x)\rangle} \cos(2x) + \cdots + \frac{\langle f(x), \sin(x)\rangle}{\langle \sin(x), \sin(x)\rangle} \sin$$

The standard is to write the above in the order of increasing the frequency of each base, hence we write

$$\begin{aligned} f(x) &= \frac{\langle f(x), \cos(0x)\rangle}{\langle \cos(0x), \cos(0x)\rangle} \cos(0x) + \frac{\langle f(x), \cos(x)\rangle}{\langle \cos(x), \cos(x)\rangle} \cos(x) + \frac{\langle f(x), \sin(x)\rangle}{\langle \sin(x), \sin(x)\rangle} \sin(x) + \frac{\langle f(x), \cos(2x)\rangle}{\langle \cos(2x), \cos(2x)\rangle} \cos( \\ &= \frac{\langle f(x), 1\rangle}{\langle 1, 1\rangle} + \frac{\langle f(x), \cos(x)\rangle}{\langle \cos(x), \cos(x)\rangle} \cos(x) + \frac{\langle f(x), \sin(x)\rangle}{\langle \sin(x), \sin(x)\rangle} \sin(x) + \frac{\langle f(x), \cos(2x)\rangle}{\langle \cos(2x), \cos(2x)\rangle} \cos(2x) + \frac{\langle f(x), \sin(}{\langle \sin(2x), \sin} \end{aligned}$$

The coordinates are given standard names: the first one is called $a_0$, the second is called $a_1$ the third is called $b_1$, etc.. i.e. the coordinates of $f(x)$ on the cos basis are called $a_0, a_1, \cdots$ and the coordinates of $f(x)$ on the sin basis are called $b_1, b_2, \cdots$. Notice that $b_0$ does not exist, since $\sin(0x) = 0$.

So, we write the above as

$$f(x) = \overbrace{\frac{\langle f(x), 1\rangle}{\langle 1, 1\rangle}}^{a_0} + \overbrace{\frac{\langle f(x), \cos(x)\rangle}{\langle \cos(x), \cos(x)\rangle}}^{a_1} \cos(x) + \overbrace{\frac{\langle f(x), \sin(x)\rangle}{\langle \sin(x), \sin(x)\rangle}}^{b_1} \sin(x) + \overbrace{\frac{\langle f(x), \cos(2x)\rangle}{\langle \cos(2x), \cos(2x)\rangle}}^{a_2} \cos(2x) + \overbrace{\frac{\langle f(x), \sin(2x}{\langle \sin(2x), \sin(2}}^{b_2}$$
$$= a_0 + a_1 \cos(x) + b_1 \sin(x) + a_2 \cos(2x) + b_2 \sin(2x) + \cdots$$

Using the above definition of an inner product, we know how to calculate each of the coordinates $a_n, b_n$ :

$$a_0 = \frac{\langle f(x), 1\rangle}{\langle 1, 1\rangle} = \frac{\int_0^{2\pi} f(x) \times 1 dx}{\int_0^{2\pi} 1 \times 1 dx} = \frac{\int_0^{2\pi} f(x)\, dx}{2\pi}$$
$$a_1 = \frac{\langle f(x), \cos(x)\rangle}{\langle \cos(x), \cos(x)\rangle} = \frac{\int_0^{2\pi} f(x) \times \cos(x)\, dx}{\int_0^{2\pi} \cos(x) \times \cos(x)\, dx} = \frac{\int_0^{2\pi} f(x) \times \cos(x)\, dx}{\pi}$$
$$a_2 = \frac{\langle f(x), \cos(2x)\rangle}{\langle \cos(2x), \cos(2x)\rangle} = \frac{\int_0^{2\pi} f(x) \times \cos(2x)\, dx}{\int_0^{2\pi} \cos(2x) \times \cos(2x)\, dx} = \frac{1}{\pi}\int_0^{2\pi} f(x) \times \cos(2x)\, dx$$

Hence we see that

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx$$
$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx) dx \qquad n > 1$$

Similarly for the $b_n$ coordinates, we obtain

$$b_1 = \frac{\langle f(x), \sin(x)\rangle}{\langle \sin(x), \sin(x)\rangle} = \frac{\int_0^{2\pi} f(x) \times \sin(x)\, dx}{\int_0^{2\pi} \sin(x) \times \sin(x)\, dx} = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(x)\, dx$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\, dx$$

We know how to measure the norm of a vector in our standard Euclidean space, so we need to decide how to measure the norm of a function in Hilbert space. For this we use the following definition

$$\| f(x)\| = \sqrt{\int_0^{2\pi} \{f(x)\}^2\, dx}$$

I used the above range of integration because for fourier series, the basis used are the $\sin(x), \cos(x)$.

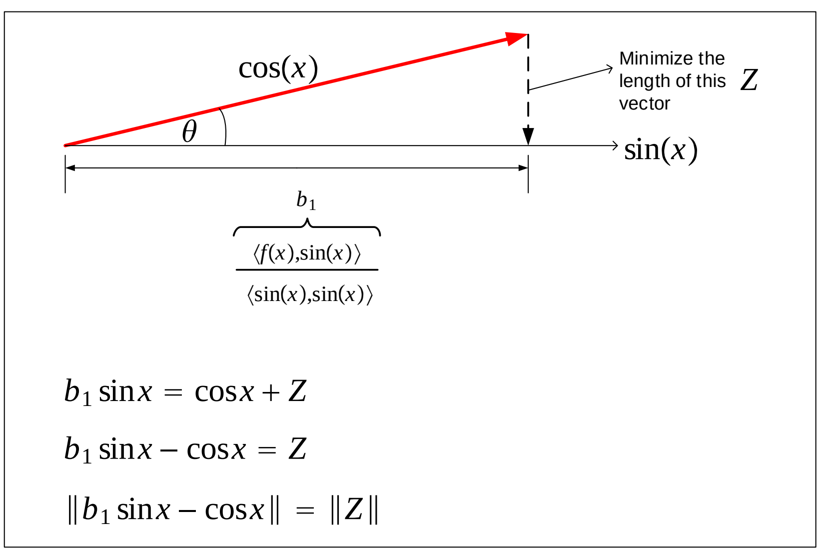
Now that we have reviewed the fourier series expansion, let us try to answer the actual question.

First, use calculus to answer the question itself:

$$
\begin{aligned}
\frac{\partial}{\partial b_1}\left(\|b_1 \sin x - \cos x\|^2\right) &= \frac{\partial}{\partial b_1}\left(\int_0^{2\pi} (b_1 \sin x - \cos x)^2\, dx\right) \\
&= \frac{\partial}{\partial b_1}\int_0^{2\pi} \left(b_1^2 \sin^2 x + \cos^2 x - 2b_1 \sin x \cos x\right) dx \\
&= \frac{\partial}{\partial b_1}\left(b_1^2 \int_0^{2\pi} \sin^2 x dx + \int_0^{2\pi} \cos^2 x dx - 2b_1 \int_0^{2\pi} \sin x \cos x dx\right) \\
&= \frac{\partial}{\partial b_1}\left(b_1^2 \left[\frac{x}{2} - \frac{1}{4}\sin 2x\right]_0^{2\pi} + \left[\frac{x}{2} + \frac{1}{4}\sin 2x\right]_0^{2\pi} - 2b_1 \left[\frac{-1}{2}\cos^2 x\right]_0^{2\pi}\right) \\
&= \frac{\partial}{\partial b_1}\left(b_1^2 [\pi] + [\pi] + b_1 \left[\cos^2 2\pi - \cos^2 0\right]\right) \\
&= \frac{\partial}{\partial b_1}\left(\pi b_1^2 + \pi\right) \\
&= 2b_1
\end{aligned}
$$

Hence for minimum, $b_1 = 0$.

Now the question is asking to compare this to the fourier coefficient $b_1$, i.e. with the coordinate $b_1$ of the function being expanded. The question did not tell us what is $f(x)$ itself. But from geometry we deduce that the problem is to minimize the distance between the function $f(x)$ and the basis, which is $\sin(x)$ in this case. Hence $b_1 \sin x - \cos x$ is the vector between the function being expressed and the basis $\sin(x)$. Hence $f(x) = \cos(x)$ in this example, as shown in this diagram



Hence, we now need to find $b_1$ given that $f(x)$ is $\cos(x)$ in this example:

$$\begin{aligned}
b_1 &= \frac{\langle f(x), \sin(x)\rangle}{\langle \sin(x), \sin(x)\rangle} \\
&= \frac{\langle \cos(x), \sin(x)\rangle}{\langle \sin(x), \sin(x)\rangle} \\
&= \frac{\int_0^{2\pi} \cos(x) \times \sin(x)\, dx}{\int_0^{2\pi} \sin(x) \times \sin(x)\, dx} \\
&= \frac{0}{\pi} \\
&= 0
\end{aligned}$$

Hence confirmed to be the same.

### 2.3.2 Problem 2

Show that the complex exponential $\phi(x) = a_0 e^{inx}$ [2] are eigen functions of the convolution operator

$$g(x) = (k * f)(x) = \int_{-\infty}^{\infty} k(x-\tau) f(\tau)\, d\tau$$

For $k \in L^2(-\infty, \infty)$ and how representing $f(x)$ as a linear combination of complex exponential greatly simplifies this equation

**Answer:** We need to show that by applying the convolution operator on $\phi(x)$, we obtain a scaled version of $\phi(x)$, i.e. need to show that

$$g(x)\rfloor_{f(x)=\phi(x)} = \lambda \phi(x)$$

Where $\lambda$ is a scalar. From the above definition, we obtain

$$\begin{aligned}
g(x) &= (k * \phi)(x) \\
&= \int_{-\infty}^{\infty} k(x-\tau)\, \phi(\tau)\, d\tau \\
&= \int_{-\infty}^{\infty} k(x-\tau)\, a_0 e^{in\tau} d\tau
\end{aligned}$$

Using the commutative property of convolution, where $(k * \phi)(x) = (\phi * k)(x)$, we can write the above as

$$\begin{aligned}
g(x) &= (\phi * k)(x) \\
&= \int_{-\infty}^{\infty} k(\tau)\, a_0 e^{in(x-\tau)} d\tau \\
&= a_0 \int_{-\infty}^{\infty} k(\tau)\, e^{inx} e^{-in\tau} d\tau \\
&= a_0 e^{inx} \int_{-\infty}^{\infty} k(\tau)\, e^{-in\tau} d\tau
\end{aligned}$$

[2] note: I renamed $e^{ikx}$ in the original question to $e^{inx}$ so as not to confuse with the $k$ function used in the definition of the convolution operator

But $\int_{-\infty}^{\infty} k(\tau) e^{-in\tau} d\tau$ is the fourier transform of the function $k(x)$, Call this fourier transform $F(k(\tau)) = X(n)$. Hence

$$\begin{aligned} g(x) &= (\phi * k)(x) \\ &= a_0 X(n)\ e^{i\omega x} \\ &= \lambda e^{i\omega x} \end{aligned}$$

Where I called $a_0 X(n)$ as the parameter $\lambda$ since $a_0 X(n)$ does not depend on $x$ but depends on $n$, I.e. given the function $k(\tau)$, we can determine its Fourier transform for the specific $n$ provided, and this Fourier transform integral, which will evaluate to some value, is multiplied with $a_0$ to obtain the scaling factor by which we scale $e^{inx}$ which is $\phi(x)$ with. Hence we showed that $\phi(x)$ is an eigenfunction of $g(x)$.

Now for the second part. If $f(x)$ can be written as linear combination of complex exponential functions as in $f(x) = \sum_{n=1}^{N} a_n e^{inx}$, then we write

$$\begin{aligned} g(x) &= (k * f)(x) \\ &= \int_{-\infty}^{\infty} k(x-\tau) f(\tau) d\tau \\ &= \int_{-\infty}^{\infty} k(\tau) f(x-\tau) d\tau \\ &= \int_{-\infty}^{\infty} k(\tau) \left( \sum_{n=1}^{N} a_n e^{in(x-\tau)} \right) d\tau \\ &= \int_{-\infty}^{\infty} k(\tau) \left( \sum_{n=1}^{N} a_n e^{inx} e^{-in\tau} \right) d\tau \\ &= \int_{-\infty}^{\infty} \left( \sum_{n=1}^{N} k(\tau) a_n e^{inx} e^{-in\tau} \right) d\tau \\ &= \sum_{n=1}^{N} \int_{-\infty}^{\infty} k(\tau) a_n e^{inx} e^{-in\tau} d\tau \\ &= \sum_{n=1}^{N} a_n e^{inx} \int_{-\infty}^{\infty} k(\tau) e^{-in\tau} d\tau \end{aligned}$$

But $\int_{-\infty}^{\infty} k(\tau) e^{-in\tau} d\tau$ is the Fourier transform of $k(\tau)$, call it $X(n)$, hence the above becomes

$$g(x) = \sum_{n=1}^{N} a_n e^{inx} X(n)$$

Hence we have replaced the integration operation with a summation operation and we have simplified this equation.

### 2.3.3 Problem 3

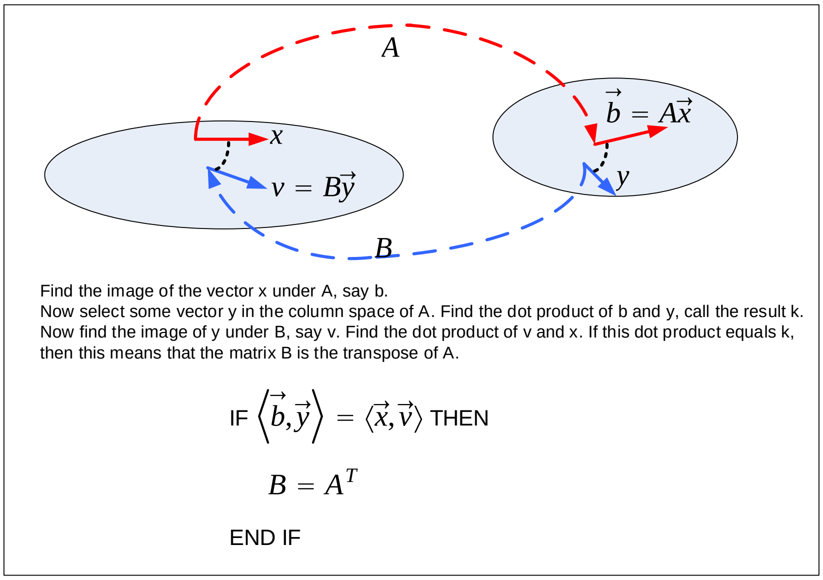
The transpose of a matrix can be defined as the matrix $A^T$ such that $\langle Ax, y \rangle = \langle x, A^T y \rangle$

This definition generalizes to function operators like the fourier transform $g(\xi) = F\{f\} = \int_{-\infty}^{\infty} f(x) e^{-i2\pi\xi x} dx$

Find the adjoint $F^T\{g\}$ using the definition above.

**Answer:**

First, a geometric view of a matrix transpose can be illustrated in this diagram




Now let us try to apply the above diagram to find the adjoint operator we need. Instead of using the Matrix notation of $A$ and $A^T$, we now use the notation of $L$ and $L^*$, where here $L^*$ is the adjoint operator of $L$. Hence we seek to find an operator $L^*$ such that $\langle Lf, q\rangle = \langle f, L^*q\rangle$

We are given what $L$ is, it is the fourier transform, it takes the function $f(x)$ and generates $g(\xi)$ according to this operation

$$g(\xi) = F\{f\} = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi\xi x} dx$$

For the inner product operation on the space of complex functions over the infinite domain, I will use the following definition

$$\langle f, g\rangle = \int_{-\infty}^{\infty} \overline{f}\, g\, dx$$

Hence, applying $\langle Lf, q\rangle = \langle f, L^*q\rangle$

$$\langle Lf, q\rangle = \langle f, L^*q\rangle$$
$$\langle g(\xi), q\rangle = \langle f, L^*q\rangle$$
$$\left\langle \overbrace{\int_{-\infty}^{\infty} f(x)\, e^{-i2\pi\xi x} dx}^{g(\xi)},\ q \right\rangle = \langle f(x), (L^*q)\rangle$$
$$\int_{-\infty}^{\infty} \left( \overline{\int_{-\infty}^{\infty} f(x)\, e^{-i2\pi\xi x} dx} \right) q\, d\xi = \int_{-\infty}^{\infty} \overline{f(x)}\, (L^*q)\, dx$$
$$\int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \overline{f(x)} e^{i2\pi\xi x} dx \right) q\, d\xi = \int_{-\infty}^{\infty} \overline{f(x)}\, (L^*q)\, dx$$

Exchanging the order of integration gives

$$\int_{-\infty}^{\infty} \overline{f(x)} \left( \int_{-\infty}^{\infty} q\, e^{i2\pi\xi x} d\xi \right) dx = \int_{-\infty}^{\infty} \overline{f(x)} \left( L^{*} q \right) dx$$

Hence we see that

$$\int_{-\infty}^{\infty} q\, e^{i2\pi\xi x} d\xi = L^{*} q$$

So, the adjoint operator is the *inverse fourier transform.*

## 2.4 HW 4

Moved to my main web page.