NASSER

SVD: $A = U\Sigma V^T$, $A_{(m\times n)} = \underset{m\times m}{U}\ \underset{m\times n}{\Sigma}\ \underset{n\times m}{V^T}$ · To start, $U \Leftrightarrow AA^T$, $V \Leftrightarrow A^TA$. Calculate AA^T and A^TA. Find which is smaller. find say AA^T is smaller. find $\bar u_1, \bar u_2$. then find σ_1^2, σ_2^2. then we have $AA^T \bar u_j = \sigma_j^2 \bar u_j$

so $A^TAA^T\bar u_j = \sigma_j^2 A^T\bar u_j \Rightarrow \boxed{\dfrac{A^T\bar u_j}{\sigma_j} = \bar v_j}$, find $\bar v_1, \bar v_2$. for $\bar v_3$ solve $(A^TA)\bar x = (0)$. this small.

if A^TA was smaller, then $\boxed{\bar u_j = \dfrac{A\bar v_j}{\sigma_j}}$ i.e. $A^TA\bar v_j = \sigma_j^2 \bar v_j \rightarrow AA^TA\bar v_j = \sigma_j^2 A\bar v_j$

so $\dfrac{A\bar v_j}{\sigma_j} = \bar u_j$. example: $\underset{2\times 3}{A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}} = \underset{2\times 2}{U}\ \underset{2\times 3}{\Sigma}\ \underset{3\times 3}{V^T}$. $U \Leftrightarrow AA^T = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $V \Leftrightarrow A^TA = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$

from U, find σ_1^2, σ_2^2, find $\bar u_1, \bar u_2$. the eigenvectors of AA^T. so now we have $[U]$.

now $\bar v_1 = \dfrac{A^T u_1}{\sigma_1}$, $\bar v_2 = \dfrac{A^T u_2}{\sigma_2}$. for $\bar v_3$: $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. so $V = \begin{pmatrix} & | & \end{pmatrix}$. Remember Now to NORmalize all eigenvalues!

$A = [u_1 | u_2 \cdots u_r \cdots u_m]\,[\sigma_1\ \sigma_2 \cdots \sigma_r \cdots]\,[v_1|v_2 \cdots v_r | \cdots v_n]^T$

(AA^T) $\quad A^TA$

similarity: $T_v = M^{-1} T_W M$ → Basis of W inverse $[\bar w_1 | \bar w_2 \cdots]^{-1}$

so $T_v \bar x_v = M^{-1} T_W M \bar x_v$

takes vector from v space to W basis

applies Linear mapping to vector in W Basis

takes vector in W basis back to v basis

$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ $a>0, ac > b^2$ test for P.D. $\bar x^T A x > 0$

Example: $(2,3)$ reflection Cross x-axis

$T(v_1) = 1\bar v_1 + 0\bar v_2 = 1\binom{1}{0} + 0\binom{0}{1}$

$T(v_2) = 0\binom{1}{0} - 1\binom{0}{1}$

$A_v = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Q: orthoNormal columns. $Q^TQ = I$

Now find new Transformation Matrix under new coordinates

$w_2\ (1,1)$ $\quad w_1\ (1,-1)$

$T(\bar w_1) = 0\bar w_1 + 1\bar w_2$

$T(\bar w_2) = 1\bar w_1 + 0\bar w_2$ $\Rightarrow A_W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Now find M:

$\binom{2}{3} = a\bar w_1 + b\bar w_2 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\binom{a}{b} \rightarrow \binom{a}{b} = \underbrace{\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1}}_{M}\binom{2}{3} \Rightarrow \binom{a}{b} = \begin{pmatrix} -1/2 \\ 5/2 \end{pmatrix}$ i.e. $M = \begin{pmatrix} \frac12 & -\frac12 \\ \frac12 & \frac12 \end{pmatrix}$. so $M^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \Rightarrow T_v = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(\ )$

To diagonalize Matrix $A = S\Lambda S^{-1}$ | $A^k = S\Lambda^k S^{-1}$. if $\Lambda = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{pmatrix} \rightarrow \Lambda^k = \begin{pmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{pmatrix}$; $e^{At} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$.

$\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ | $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ | similarity $A \xrightarrow{M^{-1}AM} B$

Inverse using Jordan. $A = \begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix} \rightarrow \left(\begin{array}{cc|cc} 2 & 0 & 1 & 0 \\ 4 & 2 & 0 & 1 \end{array}\right) \rightarrow$

$\left(\begin{array}{cc|cc} 2 & 0 & 2 & 0 \\ 0 & 2 & -2 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & \frac12 \end{array}\right)$ A^{-1}

Matrix Properties | A real & symmetric $\Rightarrow \lambda$ real.

$A = \lambda_1 x_1 x_1^T + \lambda_2 x_2 x_2^T + \cdots$

if A symm real $\Rightarrow$ spectral decomposition $A = Q\Lambda Q^{-1}$ ↳ eigenvectors orthoNormal

$Q \Rightarrow |\lambda| = 1$ orthoNormal.

if $A^HA = I \rightarrow A$ unitary.

if A is NORMAL ⟶ diagonalizable. Normal $\boxed{AA^H = A^HA}$

proof inverse of Hermitian is Hermitian: $A = A^H$

$AA^{-1} = I \rightarrow (AA^{-1})^H = I^H$

$\rightarrow (A^{-1})^H A^H = I^H$

$\rightarrow (A^{-1})^H A = I \rightarrow (A^{-1})^H = A^{-1}$

Row View $\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}(c_1) \rightarrow \begin{pmatrix} r_1 c_1 \\ r_2 c_1 \\ r_3 c_1 \end{pmatrix}$ or $\begin{pmatrix} r_1A \\ r_2A \\ r_3A \end{pmatrix}$

Col. View $\left(\begin{smallmatrix} | & | & | \\ d_1 & d_2 & d_3 \end{smallmatrix}\right)(c_i) \rightarrow (c_1 d_1 \mid c_2 d_2 \mid c_3 d_3)$

Proof different λ and A symm $\rightarrow \bar v_i$ are unique:

$(\lambda_1 x)^H y = (Ax)^H y = x^H A^H y = x^H A y = x^H \lambda_2 y$

so $\lambda_1 x^H y = \lambda_2 x^H y$. but $\lambda_1 \neq \lambda_2 \Rightarrow x^H y = 0$

i.e. $\bar x_n \perp y$

$|A^T| = |A|$

proof that if $A = A^H \rightarrow x^HAx$ real: $(x^HAx)^H = x^HA^Hx$

$= x^HAx$. i.e. $\bar z = z \Rightarrow z$ real.

$A = R^TR$ if A is symm

G.E. subtract from second row multiple of first row

proof that $A = A^H \rightarrow \lambda$ real: $Ax = \lambda x \rightarrow x^HAx = \lambda\|x\|^2$

$\rightarrow \lambda$ real since x^HAx real

Projections $\quad \bar b,\ \bar p,\ a$

$\langle b, a\rangle = \|b\|\,\|a\| \cos\theta$

$\bar p = \dfrac{\langle b,a\rangle}{\|a\|^2}\bar a = \dfrac{b^Ta}{a^Ta}\bar a$

$= \bar a\,\dfrac{a^Tb}{a^Ta} = \dfrac{\bar a\bar a^T}{a^Ta}\bar b$

$= [P]\,\bar b$ projection Matrix line on a line.

$\dfrac{du}{dt} = Au \Rightarrow \bar u = c_1 \bar v_1 e^{\lambda_1 t} + c_2 \bar v_2 e^{\lambda_2 t}$

Pure solutions $\{c_1 e^{\lambda_1 t}, c_2 e^{\lambda_2 t}\}$

projection Matrix line on plane:

$A\hat x = \underbrace{A(A^TA)^{-1}A^T}_{[P]}\,\bar b$

if $A = QR$ then $[P] = QQ^T$

$Q^TQ = I$ $\quad \boxed{\hat x = (A^TA)^{-1}A^T\bar b}$

Gram schmidt. $A = [v_1|v_2|v_3]$

$q_1 = \bar v_1$

$q_2 = v_2 - \dfrac{\langle v_2, \bar q_1\rangle}{\|q_1\|^2}\bar q_1$

$q_3 = v_3 - \left(\dfrac{\langle v_3, \bar q_1\rangle}{\|q_1\|^2}\bar q_1 + \dfrac{\langle v_3, \bar q_2\rangle}{\|q_2\|^2}\bar q_2\right)$

and Normalize!

Schwartz $|\langle a,b\rangle| \le \|a\|\,\|b\|$

Triangle $\|x+y\| \le \|x\| + \|y\|$

$A = LL^T$: A SPD

$A = QR$: A indep. cols

if A square then $Q^{-1} = Q^T$

$A = S\Lambda S^{-1}$: A has n L.I.

$A = Q\Lambda Q^T$: A symm.

$A = MJM^{-1}$: A square.

$A = U\Sigma V^T$: Any A! / $A = U\Lambda U^{-1}$: A Normal

$F_{1000} = \dfrac{1}{\sqrt5}\lambda_1^{1000}$ ↳ $\left(\dfrac{1+\sqrt5}{2}\right)$, $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$A = QR$

$R = Q^TA$

$Q^{-1} = Q^T$

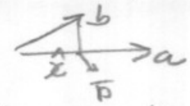
$Q^TQ = I$

Invertibility of Matrix depends on Non Zero λ

diagonalizability depends on enough eigenvectors

Solution to $u_{k+1} = Au_k$ is $u_k = A^k u_0 \Rightarrow u_k = (c_1\lambda_1^k \bar x_1 + c_2\lambda_2^k \bar x_2 + \cdots)u_0$

initial condition at $k=0$ $\quad$ Note $\lambda < 1$ for stability

Sheet Nasser  $\langle b, a\rangle = \|b\|\|a\|\cos\theta$. $\bar{P} = \frac{b^Ta}{a^Ta}\bar{a}$, $\bar{p} = \bar{a}\frac{\bar{a}^T\bar{b}}{a^Ta} = [P]\bar{b}$

$[P] = \frac{aa^T}{a^Ta}$ projection matrix line on line.

projection Matrix for line on plane is $\underbrace{A(A^TA)^{-1}A^T}_{[P]}$ so $[P]\bar{b} = \bar{P}$

if Q, then projection $[P]$ is simplified, $Q(Q^TQ)^{-1}Q^T \Rightarrow QQ^T$

so $[P] = QQ^T$ where $Q = [q_1 | q_2 \cdots]$

if basis are orthoNormal, then Projection of Vector on Plane = sum of projection of Vector on each Basis.

Note $Q^TQ = I$. Least squares: minimize $\|E\|^2 = \|Ax-b\|^2 \Rightarrow (ax_1-b_1)^2 + (ax_2-b_2)^2 + \cdots$

or using matrices, L.S. is $\hat{x} = (A^TA)^{-1}A^Tb$. error  $\bar{p} + \bar{e} = \bar{b} \Rightarrow \bar{e} = \bar{b} - \bar{p}$

$\bar{e} \perp \bar{p}$

Schwarz inequality: $\langle a, b\rangle \le \|a\|\|b\|$ or $a^Tb \le \|a\|\|b\|$

Triangle Inequality: $\|x+y\| \le \|x\| + \|y\|$, $\langle v, v\rangle = \|v\|^2$

$(x+y)^T(x+y) = x^Tx + y^Ty + 2x^Ty$

G-S: given $A = [a_1 | a_2 | \cdots]$, let $\bar{q}_1 = \bar{a}_1$

Then $\bar{q}_2 = \bar{a}_2 - \left(\frac{\bar{a}_2^Tq_1}{q_1^Tq_1}\bar{q}_1\right)$; $\bar{q}_3 = \bar{a}_3 - \left(\frac{a_3^Tq_1}{q_1^Tq_1}q_1 + \frac{a_3^Tq_2}{q_2^Tq_2}\bar{q}_2\right)$ etc..

Then Normalize $\bar{q}_i$. $A = QR \Rightarrow R = Q^TA$.

Prop of det: ① $\det I = 1$, ② row exchange: sign change, ③ Linear on first Row.

i.e $\begin{vmatrix} a+1 & b+1 \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ c & d \end{vmatrix}$; $\begin{vmatrix} 3a & 3b \\ c & d \end{vmatrix} = 3\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

if A diagonal, or triangle, $|A|$ = Product of diagonal elements. to prove, use property ③ repeatedly.

$\det(2A_n) = 2^n\det(A_n)$. $PA = LDU$. $\det P = \pm 1$.

$Qx = b$

$Q^TQ\hat{x} = Q^Tb$

$x = (Q^TQ)^{-1}Q^Tb = Q^Tb$

$\bar{P} = Q\hat{x}$

$[P] = QQ^T\bar{b}$

If $\bar{b} \perp$ to each Column of V, then $\bar{b} \perp$ span V.

if every $\bar{v} \in W \perp$ every $\bar{u} \in U$, $\Rightarrow W$ is orthogonal Complement to U

Row space $\perp$ Null space: proof: $\bar{x} \in N(A)$, $Ax = 0$.

$Ax = \begin{bmatrix} \text{---} \\ \text{---} \\ \text{---} \end{bmatrix}\begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$ each Row of $A \perp x$ since dot product $= 0$.

so every vector in Rowspace is $\perp$ every vector in $N(A)$.

similarly, $y^TA = 0$ i.e $[y_1 y_2 \cdots][\,|\,|\cdots|\,] = [0\ 0\ 0] \Rightarrow$ every y is $\perp$ every Column of A

so every vector in Left Null is $\perp C(A)$.

to find ~~Transformation~~ Matrix, apply T to each Basis. Then A = columns just found

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proof that different λ, and A symm $\Rightarrow$ eigenvectors are unique: $(\lambda_1 x)^H y = (Ax)^H y = x^H A^H y = x^H A y = x^H \lambda_2 y$.
so $\lambda_1 x^H y = \lambda_2 x^H y$, but $\lambda_1 \neq \lambda_2 \Rightarrow x^H y = 0$ i.e $\bar{x} \perp \bar{y}$.

proof that if $A = A^H \to x^H A x$ must be real: $(x^H A x)^H = x^H A^H x = x^H A x$ i.e $\bar{z} = z \Rightarrow z$ is real. i.e $x^H A x$ is real.

proof that if $A = A^H \to \lambda$ are real: $Ax = \lambda x \to x^H A x = \lambda \|x\|^2$. but $x^H A x$ is real, $\to \lambda$ is real

Given $\frac{du}{dt} = Au$, the general sol $\bar{u} = c_1 \bar{v}_1 e^{\lambda_1 t} + c_2 \bar{v}_2 e^{\lambda_2 t}$ where c_1, c_2 are B.C..

pure solutions are $\left\{ \begin{matrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{matrix} \right\}$

Prop A real symmetric Matrix can be factored into $A = Q \Lambda Q^T$. Q is matrix of orthoNormal eigenvectors

$A = \lambda_1 \underbrace{\bar{x}_1 \bar{x}_1^T}_{\text{projection}} + \lambda_2 \bar{x}_2 \bar{x}_2^T + \dots + \lambda_n \bar{x}_n x_n^T$

Spectral theorem

Matrix is uniquely determined by its λ_i and $\bar{v}_i$

$A = S \Lambda S^{-1}$

eigenvalues of Q, matrix of orthonormal eigenvectors is $|\lambda| = 1$

$A = R^T R$
if A is any symmetric

proof inverse of Hermitian is Hermitian: $A = A^H$.
$AA^{-1} = I \to (AA^{-1})^H = I^H \to (A^{-1})^H A^H = I^H \to (A^{-1})^H A = I \to (A^{-1})^H = A^{-1}$
$\to A^{-1}$ is Hermitian

$A = QR$
Q contains orthonormal Basis. $R = Q^T A$
since $Q^{-1} = Q^T$

$\boxed{Q^T Q = I}$ but Not QQ^T !

determinant properties: $|A^T| = |A|$

if we know Q, then projection Matrix $[P] = QQ^T$

AB: row indep

invertibility of Matrix depends on non zero eigenvalues
but diagonalizability depends on eigenvectors (indep).

Solution to difference equation $u_{k+1} = Au_k$ is $u_k = A^k u_0 \to$ solution $u_k = c_1 \lambda_1^k \bar{x}_1 + c_2 \lambda_2^k \bar{x}_2 + \dots$

eigenvects of A

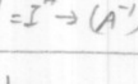
Fibonacci equation $F_{k+2} = F_{k+1} + F_k$.
To Find F_{1000} do $\frac{1}{\sqrt{5}} \lambda_1^{1000}$ when λ_1 is the bigger of λ_i
in $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$
$\lambda_1 = \frac{1+\sqrt{5}}{2}$. so $F_{1000} = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{1000}$, then round down.

Initial conditions at $k = 0$

For stability of difference equations $\lambda_i < 1$. $\lambda = 1$ neutral.
for stability of differential equation, stable if $\lambda < 0$.

$\hat{x} = (A^T A)^{-1} A^T \bar{b}$
so projection of $\hat{x}$ onto C(A) is nearest point to $A\hat{x}$
$A\hat{x} = \underbrace{A(A^T A)^{-1} A^T}_{[P] \text{ projection Matrix}} \bar{b}$
so

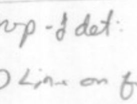
projections: vector on vector $\bar{b}$ on $\bar{a} \Rightarrow \frac{\langle b, a \rangle}{\langle a, a \rangle} \bar{a}_1$



$P^2 = P$
$P^T = P$

any symm matrix with $P^2 = P$ is projection

vector on plane: if vectors span plane, a_1, a_2, and we are asked to find projection of b on plane, then find projection of b onto a_1, and onto a_2, and add.
OR make Q matrix from a_1, a_2, and write $\bar{P} = QQ^T \bar{b}$

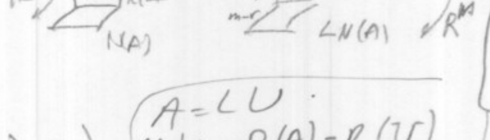


$\bar{p} + \bar{e} = \bar{b} \Rightarrow$
$\bar{e} = \bar{b} - \bar{p}$

Least square $Ax = b \Rightarrow A^T A x = A^T b \to \boxed{x = (A^T A)^{-1} A^T b}$
$\boxed{y = C + Dt}$

Schwarz inequality: $\langle a, b \rangle \leq \|a\| \|b\|$
Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$

Gram schmidt, given $\bar{v}_1, \bar{v}_2, \bar{v}_3 \to A\{v_1 | v_2 | v_3\}$
$\bar{q}_1 = \bar{v}_1$
$\bar{q}_2 = \bar{v}_2 - \left(\frac{\langle v_2, q_1 \rangle}{\|q_1\|^2} \bar{q}_1 \right)$
$\bar{q}_3 = V_3 - \left(\frac{\langle V_3, q_1 \rangle}{\|q_1\|^2} \bar{q}_1 + \frac{\langle V_3, q_2 \rangle}{\|q_2\|^2} \bar{q}_2 \right)$
Now Normalize all q's.
$A = QR$



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$\det(2A) = 2^n \det(A)$ when $A = n \times n$.

$A = LU$.
Note $R(A) = R(U)$
$N(A) = N(U)$

For Linear Transform $T(av) = aT(v)$
$T(v + x) = T(v) + T(x)$

Linear independence: $c_1 \bar{v}_1 + c_2 \bar{v}_2 \dots = \bar{0}$
$\Rightarrow$ all c's $= 0$

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