Solution key HW #8

Section 3.4 : # 1, 2, 8, 13, 16, 18
Section 4.2 # 1, 7, 14
Section 4.3 # 3 , In class problems.

Section 3.4

#1 Solution in book.

#2. $\vec{b} = (0,3,0)$ $\vec{a}_1 = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$ $\vec{q}_2 = (-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$

Since $\vec{a}_1$ and $\vec{a}_2$ are orthonormal, then the projection onto the plane spanned by $\vec{a}_1$ and $\vec{q}_2$ is the sum of the projection to each individual vector.

$$\vec{b}^T\vec{a}_1 \cdot \vec{a}_1 = 2\vec{a}_1 = \begin{bmatrix} 4/3 \\ 4/3 \\ -2/3 \end{bmatrix}$$ Projection of $\vec{b}$ onto $\vec{a}_1$

$$\vec{b}^T\vec{q}_2 \cdot \vec{a}_2 = 2\vec{a}_2 = \begin{bmatrix} -2/3 \\ 4/3 \\ 4/3 \end{bmatrix}$$ Projection of $\vec{b}$ onto $\vec{q}_2$

$$\text{Projection onto plane} = \begin{bmatrix} 4/3 \\ 4/3 \\ -2/3 \end{bmatrix} + \begin{bmatrix} -2/3 \\ 4/3 \\ 4/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 8/3 \\ 2/3 \end{bmatrix}.$$

We check by computing the projection onto the plane as the projection onto the column space of Q

$$Q = \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ -1/3 & 2/3 \end{bmatrix}$$

Since Q is ortogonal, the projection is $p = QQ^T\vec{b}$

$$= \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 8/3 \\ 2/3 \end{bmatrix} \checkmark$$

#8 $\vec{b}$ projected onto $\vec{a}_1 = \dfrac{\vec{b}^T\vec{a}_1}{\|a_1\|^2}\vec{a}_1 = \dfrac{1}{1}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$

$\vec{b} = (1,2)$ $\vec{a}_1 = (1,0)$

$\vec{b}$ projected onto $\vec{a}_2 = \dfrac{3}{2}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3/2\\3/2\end{bmatrix}$

$\vec{a}_2 = (1,1)$

see that $\begin{bmatrix}3/2\\3/2\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}5/2\\3/2\end{bmatrix} \neq \begin{bmatrix}1\\2\end{bmatrix}$

#13 $\vec{V}_1 = \begin{bmatrix}0\\0\\1\end{bmatrix}$, $\vec{V}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix} - \dfrac{1}{1}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\vec{V}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix} - \dfrac{1}{1}\begin{bmatrix}0\\0\\1\end{bmatrix} - \dfrac{1}{1}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$

I usually don't normalize until the end but in this case they are already normalized.

$A = \begin{bmatrix}0 & 0 & 1\\0 & 1 & 1\\1 & 1 & 1\end{bmatrix}$ $Q = \begin{bmatrix}0 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{bmatrix}$ $A = QR$

We find R in two ways. First slow + fundamental. one column of A at a time.

$\vec{a}_1 = 1\,\vec{V}_1$

$\vec{a}_2 = 1\,\vec{V}_1 + 1\,\vec{V}_2$ $\Rightarrow R = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}$

$\vec{a}_3 = 1\,\vec{V}_1 + 1\,\vec{V}_2 + 1\,\vec{V}_3$

or we could do the computational way $R = Q^TA$

$Q^T = \begin{bmatrix}0 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{bmatrix}$ $Q^TA = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix} = R$

#16

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ 2/3 & 1/\sqrt{2} \\ 2/3 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & \sqrt{2} \end{bmatrix}$$

$A \qquad Q \qquad R$ Using same procedure as in #13 & notes.

If we have n vectors with m components $A_{m\times n} = Q_{m\times n} R_{n\times n}$

Note that $(Q^T)_{n\times m} A_{m\times n} = R_{n\times n}$ as it should be.

#18

Q has the same column space as A, so

$$P = Q(Q^TQ)^{-1}Q^T = QQ^T.$$

or you could have used $A = QR$

$$\begin{aligned} P &= (QR)\left((QR)^T(QR)\right)^{-1}(QR)^T \\ &= (QR)(R^TQ^TQR)^{-1}R^TQ^T \\ &= (QR)(R^TR)^{-1}R^TQ^T \\ &= QRR^{-1}(R^T)^{-1}R^TQ^T \\ &= QQ^T \end{aligned}$$

Section 4.2

#1, #7 Solutions in the book.

#14

a) False; $\det \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \neq 2\det \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Linearity holds for the entire first row, not just one element

b) False; $\det \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = -1$ with pivots $1, 1$

[Row exchanges make this statement not true.]

c) False; $\underset{\text{invertible}}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} + \underset{\text{singular}}{\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}} = \underset{\text{singular}}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}$

d) True; $\det(AB) = \det A \cdot \det B = \det A \cdot 0 = 0$

e) False; $\det(AB - BA) \neq \det(AB) - \det(BA)$

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad AB = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad AB = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$AB - BA = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\det(AB - BA) = -1$!

And the assigned problems.

$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \qquad \det(A) = 5 \cdot 2 - 3 \cdot 4 = -2$

$B = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 4 & 8 & 12 \end{bmatrix} \qquad \det(B) = 1 \begin{vmatrix} 6 & 7 \\ 8 & 12 \end{vmatrix} - 2 \begin{vmatrix} 5 & 7 \\ 4 & 12 \end{vmatrix} + 3 \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix}$

$= 16 - 64 + 48 = 0$

Note that I used the cofactor expansion method but you can use other methods if you like.

$C = \begin{bmatrix} 2-\lambda & 3 \\ 4 & 5-\lambda \end{bmatrix} \qquad \det(C) = (2-\lambda)(5-\lambda) - 12$

$= \lambda^2 - 7\lambda - 2$

$D = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 5 & 6-\lambda & 7 \\ 4 & 8 & 12-\lambda \end{bmatrix} \qquad \det(D) = (1-\lambda) \begin{vmatrix} 6-\lambda & 7 \\ 8 & 12-\lambda \end{vmatrix} - 2 \begin{vmatrix} 5 & 7 \\ 4 & 12-\lambda \end{vmatrix} + 3 \begin{vmatrix} 5 & 6-\lambda \\ 4 & 8 \end{vmatrix}$

$= -\lambda^3 + 19\lambda^2 - 12\lambda$