HW # 7

Math 307

Linear Algebra, Spring 2007

CSUF

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Section 3.2 # 1, 3, 12, 18

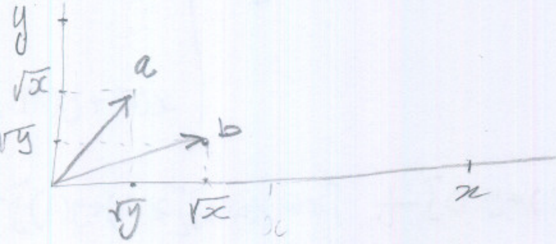
Section 3.3 # 1, 2, 3, 4, 8, 14

Section 3.2 # 1

1 (a) given any 2 positive numbers $x, y$, choose the vector $b$ equal to $(\sqrt{x}, \sqrt{y})$, and chose $a = (\sqrt{y}, \sqrt{x})$. Apply schwarz inequality to compare the arithemetic mean $\frac{1}{2}(x+y)$ with the geometric mean $\sqrt{xy}$

(b) suppose we start with a vector from the origin to the point $x$ and then add a vector of length $\|y\|$ connecting $x$ to $x+y$. the third side of the triangle goes from the origin to $x+y$. the triangle inequality asserts that this distance cannot be greater than the sum of the first two $\|x+y\| \le \|x\| + \|y\|$. reduce this to schwarz inequality

Answer (a)




$$|a^T b| \le \|a\| \, \|b\| \qquad \text{schwarz inequality}$$

i.e $\left| (\sqrt{y}, \sqrt{x}) \begin{pmatrix} \sqrt{x} \\ \sqrt{y} \end{pmatrix} \right| \le \left( \sqrt{(\sqrt{y})^2 + (\sqrt{x})^2} \right) \left( \sqrt{(\sqrt{x})^2 + (\sqrt{y})^2} \right)$

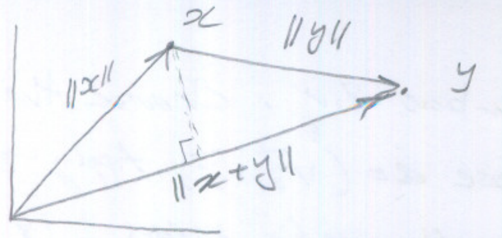
i.e $\left| \sqrt{yx} + \sqrt{xy} \right| \le \sqrt{y+x} \, \sqrt{x+y}$

i.e $2\sqrt{yx} \le y+x$

i.e $\boxed{\sqrt{yx} \le \frac{1}{2}(y+x)}$

→ back.

b)




$$\|x+y\| \leq \|x\| + \|y\|$$ square both sids:

$$\|x+y\|^2 \leq (\|x\| + \|y\|)^2 = \|x\|^2 + \|y\|^2 + 2\|x\|\,\|y\| . \quad \text{---} (1)$$

but $\|x+y\|^2 = |(x+y)^T(x+y)|$

but $|(x+y)^T(x+y)| = \{x(1)+y(1), x(2)+y(2), \ldots, x(n)+y(n)\}^T \begin{Bmatrix} x(1)+y(1) \\ x(2)+y(2) \\ \vdots \\ x(n)+y(n) \end{Bmatrix}$

$$= [x(1)+y(1)][x(1)+y(1)] + [x(2)+y(2)][x(2)+y(2)] + \cdots + [x(n)+y(n)][x(n)+y(n)]$$

$$= x^2(1) + 2x(1)y(1) + y^2(1) + x^2(2) + 2x(2)y(2) + y^2(2) + \cdots + x^2(n) + 2x(n)y(n) + y^2(n)$$

$$= [x^2(1) + x^2(2) + \cdots + x^2(n)] + [y^2(1) + y^2(2) \cdots + y^2(n)] + 2[x(1)y(1) + \cdots + x(n)y(n)]$$

so

$$\boxed{\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2x^Ty}$$ Sub this in LHS of (1)

we obtain:

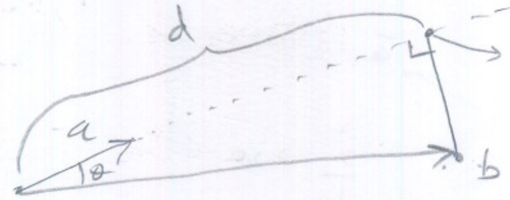
$$\|x\|^2 + \|y\|^2 + 2x^Ty \leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\|$$

$$2x^Ty \leq 2\|x\|\,\|y\|$$

$$\boxed{x^Ty \leq \|x\|\,\|y\|}$$ QED

## Section 3.2 # 3

what multiple of $a=(1,1,1)$ is closest to point $b=(2,4,4)$?
Find also the closest point to $a$ on a line through $b$.




The point is closest to $\bar{b}$ along direction of $\bar{a}$.

answer

assume we have this geometry:

so closest point along the direction of $a$ is the point when projection of $b$ onto $a$ is $90^\circ$.

but $\bar{b} \cdot \bar{a} = \|b\| \, \|a\| \cos\theta$

so $\boxed{\|b\| \cos\theta = \dfrac{\bar{b}\cdot\bar{a}}{\|a\|}}$

so to find multiples of $\bar{a}$, we write

$$\text{multiples of } a = \frac{\|b\| \cos\theta}{\|a\|} = \frac{\bar{b}\cdot\bar{a}}{\|a\|}\,\frac{1}{\|a\|}$$

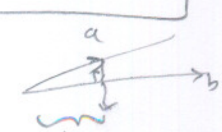
$$= \frac{\bar{b}\cdot\bar{a}}{\|a\|^2} = \frac{\bar{b}^T\bar{a}}{\|a\|^2} = \frac{[2,4,4]\begin{bmatrix}1\\1\\1\end{bmatrix}}{\left(\sqrt{1^2+1^2+1^2}\right)^2} = \frac{2+4+4}{3}$$

$$= \frac{10}{3} = 3\tfrac{1}{3}$$

so $\boxed{3\tfrac{1}{3}}$ multiples of $\|a\|$ will get to closest point to $b$

i.e the point along direction of $a$ closest to $b$ is $\boxed{\left(3\tfrac{1}{3}, 3\tfrac{1}{3}, 3\tfrac{1}{3}\right)}$

Now, to find point closest to $\bar{a}$ onto $\bar{b}$:



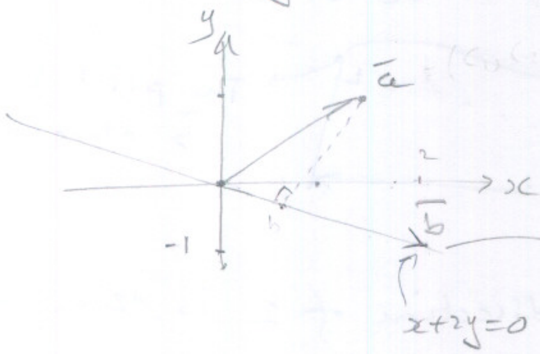

$$\|a\| \cos\theta = \frac{\bar{b}\cdot\bar{a}}{\|b\|} = \frac{10}{\sqrt{2^2+4^2+4^2}} = \frac{10}{\sqrt{36}} = \frac{10}{6}$$

unit vector along $b$

so point along $b$ is $\left(\dfrac{\bar{b}}{\|b\|}\right)\left(\|a\|\cos\theta\right) = \dfrac{\bar{b}}{\|b\|}\left(\dfrac{\bar{b}\cdot\bar{a}}{\|b\|}\right) = \dfrac{(2,4,4)}{6}\left(\dfrac{\cancel{10}^{\,5}}{\cancel{6}_{\,3}}\right) = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)\left(\frac{5}{3}\right) = \boxed{\left(\frac{5}{9}, \frac{10}{9}, \frac{10}{9}\right)}$

section 3.2 # 12

Find the matrix that projects every point in the plane onto the line $x+2y=0$




$x=-2y$

$\Rightarrow \begin{pmatrix} x \\ -\frac{1}{2}x \end{pmatrix} = c\begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}$

$\left(c, -\frac{1}{2}c\right)$

for arbitrary $c$ constant

Then $\bar{b}\cdot\bar{a} = \|a\| \, \|b\| \cos\theta$.

so $\dfrac{\bar{b}\cdot\bar{a}}{\|b\|} = \|a\| \cos\theta$.

so The projection of $\bar{a}$ onto the line $x+2y=0$ is

$$\frac{\bar{b}\cdot\bar{a}}{\|b\|}\frac{\bar{b}}{\|b\|} = \frac{b^T a}{\|b\|^2}\bar{b} = \bar{b}\,\frac{b^T a}{\|b\|^2} \Rightarrow \frac{b\,b^T}{\|b\|^2}$$

Projection Matrix

So Projection Matrix $P = \dfrac{\begin{pmatrix} c \\ -\frac{c}{2} \end{pmatrix}\left(c \;\; -\frac{c}{2}\right)}{c^2 + \frac{1}{4}c^2} = \begin{bmatrix} c^2 & -\frac{c^2}{2} \\ -\frac{c^2}{2} & \frac{c^2}{4} \end{bmatrix} \dfrac{1}{c^2\left(1+\frac{1}{4}\right)}$

$$P = \frac{4}{5}\frac{1}{c^2}\begin{bmatrix} c^2 & -\frac{c^2}{2} \\ -\frac{c^2}{2} & \frac{c^2}{4} \end{bmatrix} = \frac{4}{5}\begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

$$\begin{bmatrix} \frac{4}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix} = \boxed{\frac{1}{5}\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}}$$

note: symmetric and $P^2 = P$

Section 3.2 # 18

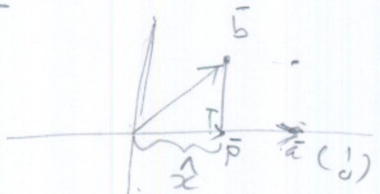
Draw the projection of b onto a and also compute it from $p = \hat{x} a$

(a) $b = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(b) $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $a \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Solution

(a)




$\bar{b} \cdot \bar{a} = \|\bar{b}\| \, \|\bar{a}\| \cos\theta$.

$$\|b\| \cos\theta = \frac{\bar{b} \cdot \bar{a}}{\|a\|}$$

unit vector along $\bar{a}$ is $\frac{\bar{a}}{\|a\|}$

hence $\bar{P} = (\|b\| \cos\theta) \frac{\bar{a}}{\|a\|} = \frac{\bar{b} \cdot \bar{a}}{\|a\|^2} \bar{a} = \frac{\bar{b}^T \bar{a}}{\|a\|^2} \bar{a}$

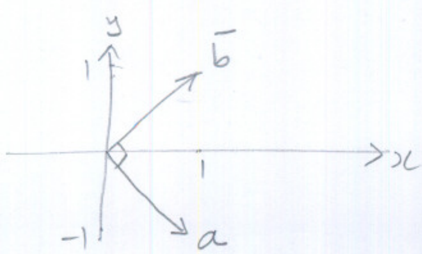
$$= \hat{x} \, \bar{a} = \frac{[\cos\theta \ \sin\theta] \begin{bmatrix} 1 \\ 0 \end{bmatrix}}{\left(\sqrt{1^2 + 0^2}\right)^2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{\cos\theta}{1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} \cos\theta \\ 0 \end{bmatrix}}$$

(b) $\bar{p} = \hat{x} \, a$

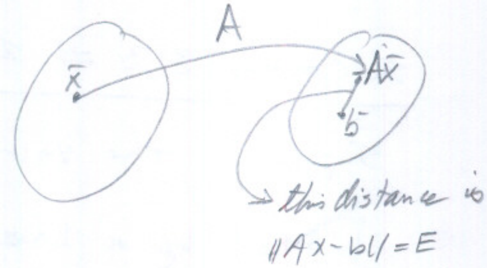
$$= \frac{[1 \ \ 1] \begin{bmatrix} 1 \\ -1 \end{bmatrix}}{\left(\sqrt{1^2 + 1^2}\right)^2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{0}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

this is because $\bar{b}$ is orthogonal to $\bar{a}$ so projection of $\bar{b}$ onto $\bar{a}$ is $\bar{0}$.



## section 3.3 #1

Find best least square solution $\hat{x}$ to $3x=10$, $4x=5$. what error $E^2$ is minimized.




### Solution

$$\boxed{A \quad \bar{x} = \bar{b}}$$

$$\left.\begin{array}{l}3x=10\\4x=5\end{array}\right\} \Longrightarrow \begin{bmatrix}3\\4\end{bmatrix} x = \begin{bmatrix}10\\5\end{bmatrix}$$

$$A\hat{x} = b$$

$$A^T A\hat{x} = A^T b$$

$$\hat{x} = (A^T A)^{-1} A^T b$$

$$= \left([3\ 4]\begin{bmatrix}3\\4\end{bmatrix}\right)^{-1} [3\ 4]\begin{bmatrix}10\\5\end{bmatrix} = [25]^{-1}(30+20) = \frac{1}{25}50 = \boxed{2}$$

so $\boxed{\hat{x} = 2}$

error being minimized is $E = \|Ax-b\| = \left\|\begin{bmatrix}3\\4\end{bmatrix}x - \begin{bmatrix}10\\5\end{bmatrix}\right\|$

$$= \left\|\begin{bmatrix}3x\\4x\end{bmatrix} - \begin{bmatrix}10\\5\end{bmatrix}\right\| = \left\|\begin{matrix}3x-10\\4x-5\end{matrix}\right\| = \sqrt{(3x-10)^2+(4x-5)^2}$$

so $\boxed{E^2 = (3x-10)^2 + (4x-5)^2}$

we minimize $E^2$ above.

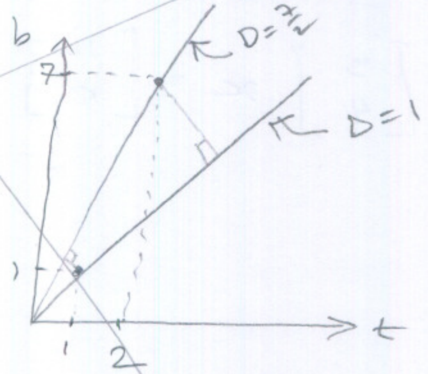
to check that error vector $\begin{pmatrix}10-3\hat{x}\\5-4\hat{x}\end{pmatrix}$ is $\perp$ to $\begin{pmatrix}3\\4\end{pmatrix}$, we take dot product and see if zero.

$$\begin{pmatrix}10-3(2)\\5-4(2)\end{pmatrix}\cdot\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}10-6\\5-8\end{pmatrix}\cdot\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}4\\-3\end{pmatrix}\cdot\begin{pmatrix}3\\4\end{pmatrix}$$

$$= (4)(3) + (-3)(4) = 0 \qquad \boxed{\text{yes.}}$$

## section 3.3 # 2

Suppose the values $b_1=1$, $b_2=7$ at times $t_1=1$, $t_2=2$ are fitted by a line $b=Dt$ through the origin. Solve $D=1$, and $2D=7$ by least squares. and sketch the best line.




Answer

opps! Please ignore →

in first Case, $D=1$

Find $|E|^2$ and minimize. let $E_1$ be error from ~~(1,1) to~~ line $D=1$, and let $E_2$ be error from $(2,7)$ to line $D=2$.

From diagram on right, in this Case, we have


$\bar{e}=\bar{b}-\bar{P}$

$\bar{e}=\bar{b}-\left(\frac{\bar{b}\cdot\bar{a}}{\|a\|^2}\right)\bar{a}$

$$\bar{b}=(1,1)$$
$$\bar{a}=(1,1)$$

so $\bar{e}_1=(1,1)-\left(\frac{(1,1)\binom{1}{1}}{\left(\sqrt{1^2+1^2}\right)^2}\right)(1,1)$

$=(1,1)-\left(\frac{2}{2}\right)(1,1)=0$ so $\boxed{|E_1|=0.}$

for $\bar{e}_2$, this is error between Point $(2,7)$ to line $D=1$

hence, $\bar{b}=(2,7)$, $\bar{a}=(1,1)$.

so $\bar{e}_2=(2,7)-\left(\frac{(2,7)\binom{1}{1}}{2}\right)(1,1)=(2,7)-\left(\frac{9}{2}\right)(1,1)$

$=(2,7)-\left(\frac{9}{2},\frac{9}{2}\right)=\left(2-\frac{9}{2},\ 7-\frac{9}{2}\right)=\left(\frac{4-9}{2},\frac{14-9}{2}\right)$

$\bar{e}_2=\left(-\frac{5}{2},\frac{5}{2}\right)$. so $|E_2|=\sqrt{\frac{25}{4}+\frac{25}{4}}=\sqrt{\frac{50}{4}}=\boxed{\frac{5}{2}\sqrt{2}}$

so $|E|=\frac{5}{2}\sqrt{2}$ in the Case fitting is done using $D=1$

i.e. $|E|^2=\frac{25}{2}=\boxed{12.5}$

For $D=\frac{7}{2}$ case →

Section 3.3 # 2

Fit line through $(1,1)$, $(2,7)$ using least squares.

Solution    equation of line is $\boxed{b = C - Dt}$

write equations as if line goes through points:

Point $(1,1)$    $b=1,\ t=1 \Rightarrow 1 = C + D$

Point $(2,7)$    $b=7,\ t=2 \Rightarrow 7 = C + 2D$

$$\left.\begin{array}{l} 1 = C + D \\ 7 = C + 2D \end{array}\right\} \underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}} \underset{x}{\begin{bmatrix} C \\ D \end{bmatrix}} = \underset{b}{\begin{bmatrix} 1 \\ 7 \end{bmatrix}}$$

Solve by Least squares:

$$A^T A \hat{x} = A^T b$$

$$\hat{x} = (A^T A)^{-1} A^T b$$

but $A^T = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ so $A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ so $(A^T A)^{-1} = \frac{1}{10-9}\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$

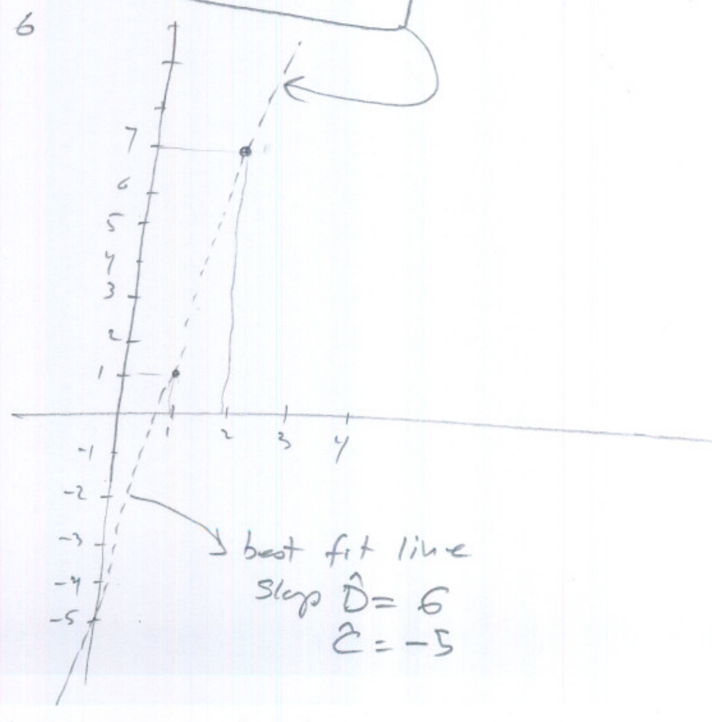
so $\hat{x} = \underset{2\times2}{\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}} \underset{2\times2}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}} \underset{2\times1}{\begin{bmatrix} 1 \\ 7 \end{bmatrix}} = \underset{2\times2}{\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}} \underset{2\times1}{\begin{bmatrix} 1 \\ 7 \end{bmatrix}} = \begin{bmatrix} -5 \\ -6 \end{bmatrix}$

so $\hat{C} = -5$, $\hat{D} = -6$.

So best fit is line $\boxed{b = -5 + 6t}$

@ $b=1 \Rightarrow t=1$
@ $b=7 \Rightarrow t=2$

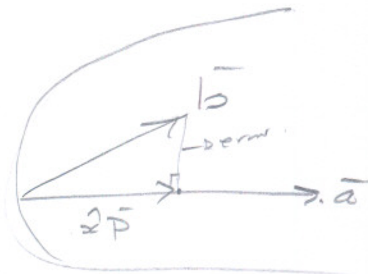
intersect $b=-5$, slop $= 6$



section 3.3 #3

Solve $Ax=b$ by least squares and find $p=A\hat{x}$. Verify the error $b-p$ is $\perp$ the columns of $A$

$$A=\begin{bmatrix}1&0\\0&1\\1&1\end{bmatrix},\quad b=\begin{bmatrix}1\\1\\0\end{bmatrix}$$




## Answer

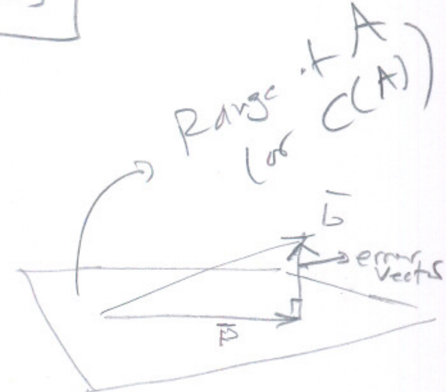
$$A^TA\,\hat{x}=A^T\bar{b}$$

$$\hat{x}=(A^TA)^{-1}A^T\bar{b}$$

$$=\left(\begin{bmatrix}1&0&1\\0&1&1\end{bmatrix}\begin{bmatrix}1&0\\0&1\\1&1\end{bmatrix}\right)^{-1}\begin{bmatrix}1&0&1\\0&1&1\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}2&1\\1&2\end{bmatrix}^{-1}\begin{bmatrix}1\\1\end{bmatrix}$$

$$=\frac{1}{4-1}\begin{bmatrix}2&-1\\-1&2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\frac{1}{3}\begin{bmatrix}1\\1\end{bmatrix}\Rightarrow\boxed{\hat{x}=\begin{bmatrix}1/3\\1/3\end{bmatrix}}$$

$$\text{so }\bar{p}=\begin{bmatrix}1&0\\0&1\\1&1\end{bmatrix}\begin{bmatrix}1/3\\1/3\end{bmatrix}=\boxed{\begin{bmatrix}1/3\\1/3\\2/3\end{bmatrix}}$$




to verify error $\bar{b}-\bar{p}$ is $\perp$ to columns of $A$:

$$\bar{b}-\bar{p}=\begin{bmatrix}1\\1\\0\end{bmatrix}-\begin{bmatrix}1/3\\1/3\\2/3\end{bmatrix}=\boxed{\begin{bmatrix}2/3\\2/3\\-2/3\end{bmatrix}}$$

$\bar{p}+\bar{e}=\bar{b}\Rightarrow\bar{e}=\bar{b}-\bar{p}$

to verify $\bar{e}\perp$ columns of $A$, take dot product with each col.

$$\left.\begin{aligned}e^T\begin{bmatrix}1\\0\\1\end{bmatrix}&=\begin{bmatrix}2/3&2/3&-2/3\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix}=\frac{2}{3}-\frac{2}{3}=0\\ e^T\begin{bmatrix}0\\1\\1\end{bmatrix}&=\begin{bmatrix}2/3&\frac{2}{3}&-\frac{2}{3}\end{bmatrix}\begin{bmatrix}0\\1\\1\end{bmatrix}=\frac{2}{3}-\frac{2}{3}=0\end{aligned}\right\}\text{ Verified ok.}$$

section 3.3 # 4

$A=\begin{bmatrix}1&0\\0&1\\1&1\end{bmatrix}$, $x=\begin{bmatrix}u\\v\end{bmatrix}$, $b=\begin{bmatrix}1\\3\\4\end{bmatrix}$. Find $\hat{x}$ using method of $E^2=\|Ax-b\|^2$ and minimize. and also by method of $A^TA\hat{x}=A^Tb$. why is $\bar{p}=\bar{b}$ ?

Answer

$$A=\begin{bmatrix}1&0\\0&1\\1&1\end{bmatrix},\ x=\begin{bmatrix}u\\v\end{bmatrix},\ b=\begin{bmatrix}1\\3\\4\end{bmatrix}$$

$$E^2=\|Ax-b\|^2=\Big\|\underbrace{\begin{bmatrix}1&0\\0&1\\1&1\end{bmatrix}}_{3\times2}\underbrace{\begin{bmatrix}u\\v\end{bmatrix}}_{2\times1}-\begin{bmatrix}1\\3\\4\end{bmatrix}\Big\|^2$$

$$=\Big\|\begin{bmatrix}u\\v\\u+v\end{bmatrix}-\begin{bmatrix}1\\3\\4\end{bmatrix}\Big\|^2$$

$$E^2=\Big\|\begin{matrix}u-1\\v-3\\u+v-4\end{matrix}\Big\|^2=(u-1)^2+(v-3)^2+(u+v-4)^2$$

$$\frac{d(E^2)}{du}=0\Rightarrow 2(u-1)+2(u+v-4)=2u-2+2u+2v-8=0$$
$$4u+2v-10=0$$

$$\frac{d(E^2)}{dv}=0\Rightarrow 2(v-3)+2(u+v-4)=2v-6+2u+2v-8=0$$
$$4v+2u-14=0$$

so $\left.\begin{matrix}4u+2v-10=0\\4v+2u-14=0\end{matrix}\right\}$ $u=\dfrac{10-2v}{4}$

$$4v+2\left(\frac{10-2v}{4}\right)-14=0\Rightarrow 4v+5-v-14=0\Rightarrow 3v=9$$
$$\boxed{v=3}$$

so $u=\dfrac{10-2(3)}{4}=\dfrac{10-6}{4}=\boxed{1}$

so $\boxed{u=1,\ v=3}$ this gives $\dfrac{d(E^2)}{du}=0$ and $\dfrac{d(E^2)}{dv}=0$.

so $\hat{x}=\begin{bmatrix}1\\3\end{bmatrix}$ that minimizes error in least square sense.

now find $\hat{x}$ using $A^TA\hat{x}=A^Tb$:

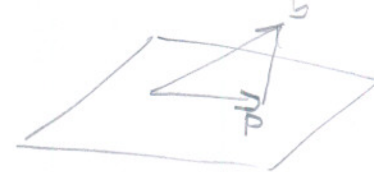
$$\hat{x}=(A^TA)^{-1}A^Tb=\left(\begin{bmatrix}1&0&1\\0&1&1\end{bmatrix}\begin{bmatrix}1&0\\0&1\\1&1\end{bmatrix}\right)^{-1}\begin{bmatrix}1&0&1\\0&1&1\end{bmatrix}\begin{bmatrix}1\\3\\4\end{bmatrix}=\begin{bmatrix}2&1\\1&2\end{bmatrix}^{-1}\begin{bmatrix}5\\7\end{bmatrix}$$

$$=\frac{1}{3}\begin{bmatrix}2&-1\\-1&2\end{bmatrix}\begin{bmatrix}5\\7\end{bmatrix}=\frac{1}{3}\begin{bmatrix}3\\9\end{bmatrix}=\begin{bmatrix}1\\3\end{bmatrix}$$ which matches method using calculus

$$\hat{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

$$\bar{P} = A\hat{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

note : $\bar{P} = \bar{b}$.




notice that $\bar{P} = \bar{b}$ in this example.

The reason is that $\bar{b}$ Lies in $C(A)$.

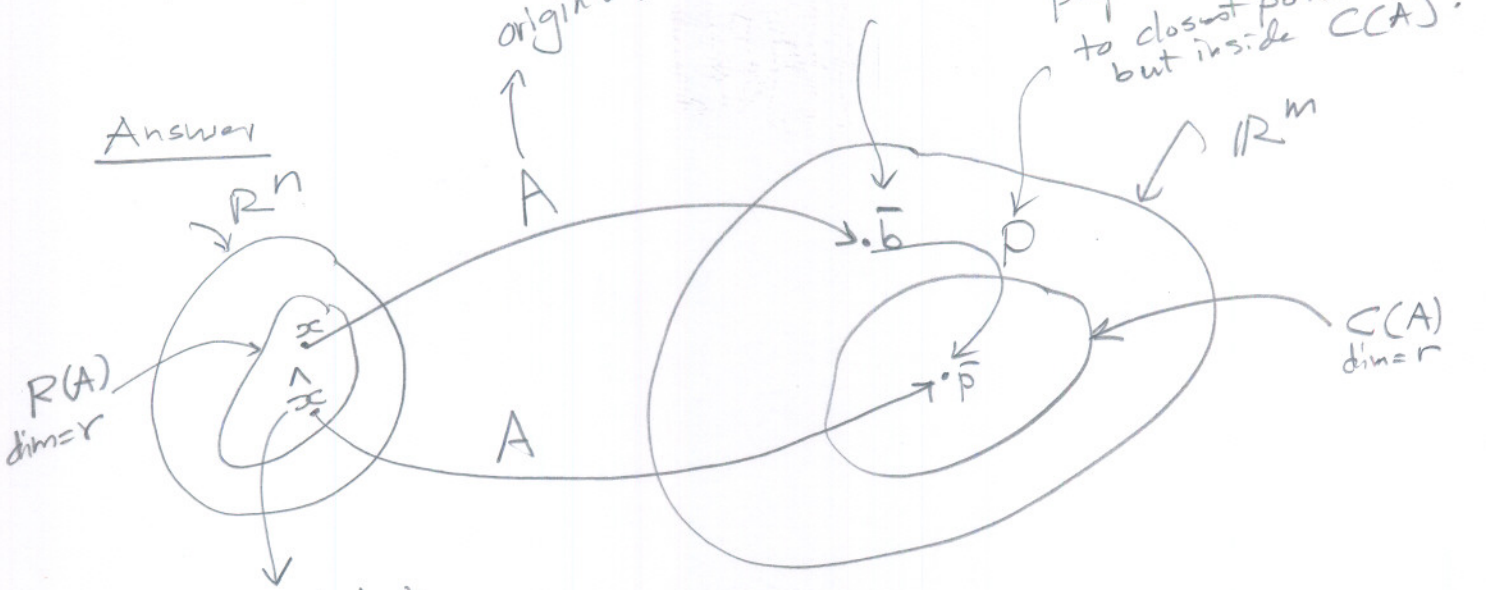
3 times 1st col of A + 1 times 2nd col A = $\bar{b}$.

so $\bar{b}$ is on the "plane" of $C(A)$. hence error is zero. so $\bar{b} = \bar{P}$.

Section 3.3 #8

If $P$ is projection Matrix onto the $K$ dimensional subspace of $R^n$, find $C(P)$ and its rank.

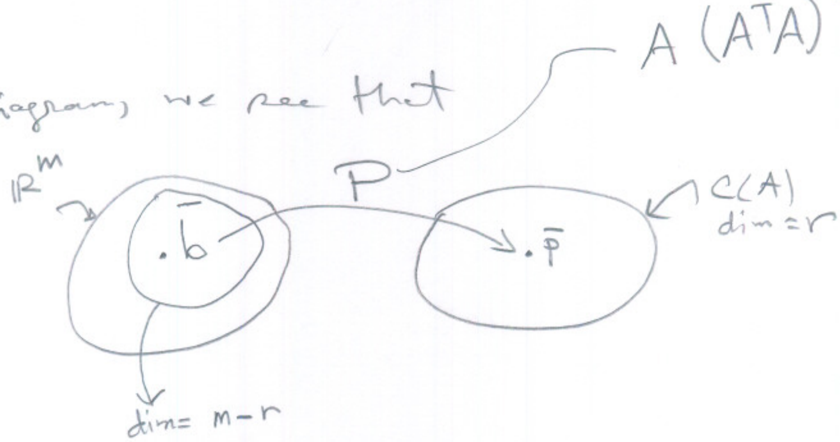
Answer




this is least squares error solution.

best approximation to solution that gives $\bar{p}$ which is closest to $\bar{b}$. i.e $A\hat{x}=\bar{p}$

from above diagram, we see that




so $\boxed{\text{Column space of } P \text{ is } C(A).}$ we are told this has dim=K

so $r=K$.

its rank = $m-r = m-K$

if $A$ is square, then $P$ rank = $\boxed{n-K} = \boxed{m-K}$   $A = m\times n$

Section 3.3 # 14

The vectors $a_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}$, $a_2 = \begin{pmatrix}1\\1\\1\end{pmatrix}$ span a plane in $\mathbb{R}^3$. find the projection matrix $P$ onto the plane and a nonzero vector $\underline{b}$ that is projected to zero.

Answer

$$A = \begin{bmatrix}1 & 1\\1 & 1\\0 & 1\end{bmatrix}$$

$$P = A(A^TA)^{-1}A^T = \begin{bmatrix}1 & 1\\1 & 1\\0 & 1\end{bmatrix}\left(\begin{bmatrix}1 & 1 & 0\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 1\\0 & 1\end{bmatrix}\right)^{-1}\begin{bmatrix}1 & 1 & 0\\1 & 1 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 1\\1 & 1\\0 & 1\end{bmatrix}\left(\begin{bmatrix}2 & 2\\2 & 3\end{bmatrix}\right)^{-1}\begin{bmatrix}1 & 1 & 0\\1 & 1 & 1\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & 1\\0 & 1\end{bmatrix}\frac{\begin{bmatrix}3 & -2\\-2 & 2\end{bmatrix}}{6-4}\begin{bmatrix}1 & 1 & 0\\1 & 1 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 1\\1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}3/2 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\1 & 1 & 1\end{bmatrix} = \begin{bmatrix}1/2 & 0\\1/2 & 0\\-1 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\1 & 1 & 1\end{bmatrix} = \boxed{\begin{bmatrix}1/2 & 1/2 & 0\\1/2 & 1/2 & 0\\0 & 0 & 1\end{bmatrix}}$$

now solve for $P\underline{b} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

so $$\begin{bmatrix}1/2 & 1/2 & 0\\1/2 & 1/2 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

so $b_3 = 0$

$\frac{1}{2}b_1 + \frac{1}{2}b_2 = 0 \Rightarrow b_1 = -b_2 \Rightarrow \bar{b} = \begin{bmatrix}b_1\\-b_1\\0\end{bmatrix}$

$$\boxed{\bar{b} = t\begin{bmatrix}1\\-1\\0\end{bmatrix}}$$

so $\bar{b}$ is vector through point $(1,-1,0)$ For $t=1$

it is $\boxed{\begin{bmatrix}1\\-1\\0\end{bmatrix}}$