~~19~~ 20/20 ~~nice~~ Great!

Math 307

CSUF

winter/spring 2007.

HW # ~~6~~ 5

Section 2.3 # 1, 5, 11, 19, 20, 34, 38

Section 2.4 # 2, 4, 6, 14, 15, 25, 37

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Time : ~ 15 hrs

section 2.3 # 1

Show that v_1, v_2, v_3 are independent but v_1, v_2, v_3, v_4 are dependent

$$v_1 = \begin{bmatrix}1\\0\\0\end{bmatrix},\ v_2 = \begin{bmatrix}1\\1\\0\end{bmatrix},\ v_3 = \begin{bmatrix}1\\1\\1\end{bmatrix},\ v_4 = \begin{bmatrix}2\\3\\4\end{bmatrix}$$

Solve $c_1 v_1 + \cdots + c_4 v_4 = 0$ or $Ac = 0$

The v's go in the columns of A.

Solution

to show that v_1, v_2, v_3 are independent, need to show that

$$c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 = \bar{0} \quad \text{only for } c_1 = c_2 = c_3 = 0$$

hence $c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}1\\1\\0\end{bmatrix} + c_3\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$.

Solve for $\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}$. already in U form. hence apply back sub:

$$\left.\begin{aligned}(1)c_3 = 0 &\Rightarrow c_3 = 0\\ c_2 + c_3 = 0 &\Rightarrow c_2 = 0\\ c_1 + c_2 + c_3 = 0 &\Rightarrow c_1 = 0\end{aligned}\right\} \text{ hence only solution is } \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

hence $\boxed{L.I.}$

To show if v_1, v_2, v_3, v_4 are L.I or NOT, repeat as above.

$$c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}1\\1\\0\end{bmatrix} + c_3\begin{bmatrix}1\\1\\1\end{bmatrix} + c_4\begin{bmatrix}2\\3\\4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}\boxed{1}&1&1&2\\0&\boxed{1}&1&3\\0&0&1&4\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\\c_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \begin{array}{l}\text{pivot variables } c_1, c_2\\ \text{free variables } c_3, c_4\end{array}$$

$$\Rightarrow \left.\begin{aligned}c_1 + c_2 + c_3 + 2c_4 &= 0\\ c_2 + c_3 + 3c_4 &= 0\\ c_3 + 4c_4 &= 0\end{aligned}\right\} \Rightarrow \begin{aligned}c_1 &= -c_2 - c_3 - 2c_4 = c_3 + 3c_4 - c_3 - 2c_4 = c_4\\ c_2 &= -c_3 - 3c_4 = 4c_4 - 3c_4 = c_4\\ c_3 &= -4c_4\end{aligned} \Bigg\} \Rightarrow c = c_4\begin{bmatrix}1\\1\\-4\\1\end{bmatrix}$$

$\longrightarrow$

therefor, we can find the following combination of c's to give zero where c's are NOT all zero:

let $c_4 = 1$. $\Rightarrow$ $c_1 = 1$, $c_2 = 1$, $c_3 = -4$, $c_4 = 1$

try it:

$$1 \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \times \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1 \times \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

hence $\boxed{v_1 + v_2 - 4v_3 + v_4 = 0}$

So we found combination that gives zero, even though the c's are not all zero $\Rightarrow$ Linearly dependent.

section 2.3 # 5

decide on dependence or independence of

a) $v_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, v_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$

$$\Rightarrow c_1 \bar{v}_1 + c_2 \bar{v}_2 + c_3 \bar{v}_3 = \bar{0} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

✓

$$\xrightarrow[\ell_{31}=2]{\ell_{21}=3} \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{-5} & -7 \\ 0 & -1 & -5 \end{bmatrix} \xrightarrow{\ell_{32}=\frac{1}{5}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & -\frac{18}{5} \end{bmatrix} \Rightarrow \text{since Full Rank} \Rightarrow \boxed{L.I.}$$

↑ ↑ ↑

b) $v_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, v_3 = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$

$$\Rightarrow c_1 \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + c_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -3 & 1 \end{bmatrix}$$

✓

$$\xrightarrow[\ell_{31}=2]{\ell_{21}=-3} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & -7 & 7 \end{bmatrix} \xrightarrow{\ell_{32}=-1} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Rank} < \text{Number Columns}$$

$$\Rightarrow \boxed{L.D.}$$

Section 2.3 # 11

describe subspace $\mathbb{R}^3$ spanned by

(a) 2 vectors $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$.

since These 2 vectors are L.D. (add them we get Zero), then we really have one Vector to use as Basis. <u>one basis</u> can only span a <u>Line</u>.

(b) $\underbrace{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

These 2 are L.I. look at $v(1)$ entry in each vector. There is no combination $c_1(0) + c_2(1) = 0$ other than $c_1 = 0$, $c_2 = 0$.

since we have 2 Basis, we can span a <u>plane</u>

(c) $\begin{bmatrix} \boxed{x} & x & x & x & x \\ 0 & \boxed{x} & x & x & x \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

since 5 columns, the solution lives in $\mathbb{R}^5$

since 3 Rows, then $\underline{b}$ lives in $\mathbb{R}^3$.

since 2 pivots, then $C(A)$ spans $\mathbb{R}^2$, which is subspace in $\mathbb{R}^3$. Therefore, $C(A)$ is a <u>plane</u> in $\mathbb{R}^3$.

Section 2.3 # 11

(d) all vectors with positive components.

since we have all vectors with positive components, then among these we can find the 3 Basis $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$.

~~therefor~~ these can span $\subset \mathbb{R}^3$ s.t. each vector is of form $\begin{Bmatrix}x_1\\x_2\\x_3\end{Bmatrix}$ where $x_i > 0$ $i = 1, 2, 3$.

in $\mathbb{R}^2$ For example: all Vectors with positive components are like this:

so they span the 1st quadrant. (This is plane in $\mathbb{R}^2$).

in $\mathbb{R}^3$ For example

vectors with +ve components live here.

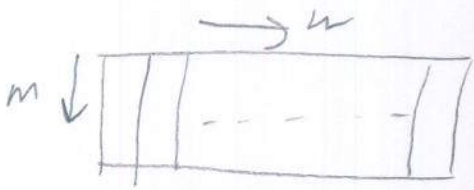
This is "3D Volume" space in $\mathbb{R}^3$)

i.e. 3D space, but subspace of $\mathbb{R}^3$.

Section 2.3 # 19

if $v_1, \ldots, v_n$ are Linearly independent, the space they span has dimension $\underline{n}$. These vectors are a $\underline{\text{Basis}}$ for that space. If the vectors are the Columns of an $m \times n$ matrix, then m is $\underline{>=}$ than n.

Further discussion:



since we have n L.I. Columns, then we have n <u>pivots</u>. Therefor m, <u>Can NOT</u> be less than n. since <u>each row</u> can have at max one pivot on it. hence m must be at least as Large as n.

it can be Larger than n, which means we have more equations than unknowns. (overdetermined system)

Section 2.3 # 20

Find Basis for each of these subspace in $\mathbb{R}^4$.

(a) all vector whose components are equal

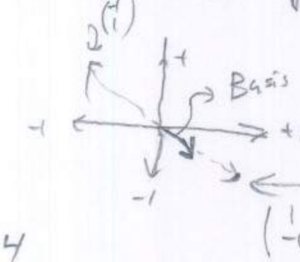
since all vectors have equal components, then they are L.D. to each others. so pick ONE of these as our only Basis. So the Basis is $\begin{pmatrix} c \\ c \\ c \\ c \end{pmatrix}$ where c is the Component.

i.e the basis is a vector in $\mathbb{R}^4$. (line) sub space is a

(b) All vectors whose Components add to zero.

in $\mathbb{R}^2$, this is like

so Basis is $\boxed{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$



So it is a line subspace

possible Combination $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

by extension, in $\mathbb{R}^4$,

possible combinations are

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

So these are the Basis. $\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$ they span $\mathbb{R}^3$ in $\mathbb{R}^4$

section 2.3 # 20

(c) find basis for subspace in R^4 where all vectors are $\perp$ to $\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\1\end{pmatrix}$

answer.

a vector $\underline{a}$ is $\perp$ to $\underline{b}$ if $\underline{a}\cdot\underline{b} = 0$.

Let $\underline{v} = \begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix}$. so we have $(v_1\ v_2\ v_3\ v_4)^T \cdot \begin{pmatrix}1\\1\\0\\0\end{pmatrix} = 0$ —— (1)

and $(v_1\ v_2\ v_3\ v_4)^T \cdot \begin{pmatrix}1\\0\\1\\1\end{pmatrix} = 0$ —— (2)

from (1) we have: $v_1 + v_2 = 0 \Rightarrow v_1 = -v_2$

from (2) we have: $v_1 + v_3 + v_4 = 0 \Rightarrow v_3 = -v_1 - v_4$

so $\underline{v} = \begin{bmatrix}v_1\\-v_1\\-v_1-v_4\\v_4\end{bmatrix} = v_1 \begin{bmatrix}1\\-1\\-1\\0\end{bmatrix} + v_4 \begin{bmatrix}0\\0\\-1\\1\end{bmatrix}$

therefore bases for subspace is $\boxed{\left\{ \begin{bmatrix}1\\-1\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\-1\\1\end{bmatrix} \right\}}$

(d) The Column space in $\mathbb{R}^2$ and null space in $\mathbb{R}^5$ of $U = \begin{bmatrix}\boxed{1} & 0 & 1 & 0 & 1\\ 0 & \boxed{1} & 0 & 1 & 0\end{bmatrix}$

$C(U)$ Basis: $\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$

$N(U)$ $Ax=0 \Rightarrow \begin{bmatrix}\boxed{1} & 0 & 1 & 0 & 1\\ 0 & \boxed{1} & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \Rightarrow$ $\begin{aligned}x_1 + x_3 + x_5 &= 0\\ x_2 + x_4 &= 0\end{aligned}$

pivot variables $= x_1, x_2$, free variables x_3, x_4, x_5

$x_1 = -x_3 - x_5$
$x_2 = -x_4$
$\Rightarrow \underline{x} = \begin{bmatrix}-x_3-x_5\\-x_4\\x_3\\x_4\\x_5\end{bmatrix} = x_3\begin{bmatrix}-1\\0\\1\\0\\0\end{bmatrix} + x_4\begin{bmatrix}0\\-1\\0\\1\\0\end{bmatrix} + x_5\begin{bmatrix}-1\\0\\0\\0\\1\end{bmatrix}$.

so $N(U)$ Basis: $\boxed{\left\{ \begin{bmatrix}-1\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\0\\1\end{bmatrix} \right\}}$

section 2.3 # 34

Prove that if $\underline{V}$ and $\underline{W}$ are 3D subspace of R^5, then $\underline{V}$ and W must have a nonzero vector in common.

$\mathbb{R}^5$ has 5 Basis since space is of dimension 5.

let these basis be called $\{r_1, r_2, r_3, r_4, r_5\} = \boxed{S_1}$

$\underline{V}$ is $\mathbb{R}^3$ subspace of $\mathbb{R}^5$, therefor its basis must be some 3 basis out of the basis of $\mathbb{R}^5$. pick any 3 in random out of basis for $\mathbb{R}^5$. say $\{r_1, r_2, r_3\} = S_2$. Call this $\boxed{S_2}$

we see that $\underbrace{S_2}_{\underline{V}\text{ Basis}} \subset \overbrace{S_1}^{R5\text{ Basis}}$.

Now for W, since it is $\mathbb{R}^3$ also, so pick another 3 Basis out of S_1 to use as basis for W. there are only 2 basis left in S_1 that are Not in S_2. hence to have 3 basis in W we must pick one basis which is 'shared' with V.

So there is one common vector between V, W.

Section 2.3 # 38

(a) Find all functions that satisfy $\frac{dy}{dx} = 0$.

since equation of a line is $ax + by = d$ (in 2D)

$\Rightarrow y = \frac{d - ax}{b}$ or $\boxed{y = \frac{d}{b} - \frac{a}{b}x}$ $\rightarrow \boxed{\frac{dy}{dx} = -\frac{a}{b}}$

for $\frac{dy}{dx}$ to equal zero, then we must have $\frac{a}{b} = 0$, or $a = 0$.

so $\boxed{y(x) = \text{Constant}}$ or $y = \frac{d}{b} = \text{Constant}$

(b) choose a particular function that satisfy $\frac{dy}{dx} = 3$.

So $y(x) = \int 3\, dx = 3x + K$ → integration Constant.

So $\boxed{y(x) = K + 3x}$ by selecting a particular value of K we get:

$\boxed{y(x) = K_1 + 3x}$ where K_1 is some constant

(C) all function that satisfy $\frac{dy}{dx} = 3$ are of the form $\boxed{y(x) = K + 3x}$ where K is a Constant.

Section 2.4 # 2 Not the same problem in the U.S. edition, please check that the problems are the same.

find the dimensions and a basis for the 4 Fundamental Subspaces for

$$\underset{m\times n = 3\times 4}{A} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Answer

need to find $\boxed{C(A),\ N(A),\ R(A),\ \text{left Null}(A).}$

for Matrix A

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \xrightarrow{l_{31}=1} \begin{bmatrix} \boxed{1} & 2 & 0 & 1 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so column 1, and Column 2 are L.I.

hence $\boxed{C(A) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\}}$

$C(A) \subseteq \mathbb{R}^m$ i.e. $C(A)$ in $\mathbb{R}^3$ rank.

subspace $C(A)$ dimension is $r = 2$

i.e. $\boxed{C(A) \text{ is plane in } \mathbb{R}^3.}$

to find $N(A)$, solve $Ax = 0$.

$$\left.\begin{aligned} x_1 + 2x_2 + x_4 &= 0 \\ x_2 + x_3 &= 0 \\ x_1 + 2x_2 + x_4 &= 0 \end{aligned}\right\}$$

pivot variables $= x_1, x_2$

free variables $= x_3, x_4$.

so $x_1 = -2x_2 - x_4 = -2(-x_3) - x_4 = 2x_3 - x_4$

$x_2 = -x_3$

$$\Rightarrow \underline{X}_n = \begin{bmatrix} 2x_3 - x_4 \\ -x_3 \\ x_3 \\ x_4 \end{bmatrix}$$

so $\underline{X}_n = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \boxed{N(A) = \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}}$

$N(A)$ in $\mathbb{R}^n$. dimension of $N(A) = \underset{n}{4} - r = 2$ i.e. $\boxed{N(A) \text{ is a plane in } \mathbb{R}^4}$

$R(A) = C(A^T)$. but $A^T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

So $\boxed{R(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}}$

$R(A) \subseteq \mathbb{R}^4$, since $r = 2$, the dimension of $R(A) = 2$

i.e. $\boxed{R(A) \text{ is plane in } \mathbb{R}^4}$.

left Null(A) $\longrightarrow$

Left Null (A) is Null space of A^T.

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{array}{l} \text{pivot variables} = x_1, x_2 \\ \text{free variables} = x_3 . \end{array}$$

$$\Rightarrow A^T x = 0 \Rightarrow \begin{array}{l} x_1 + x_3 = 0 \\ 2x_1 + x_2 + 2x_3 = 0 \\ x_2 = 0 \\ x_1 + x_3 = 0 \end{array} \rightarrow x_1 = -x_3$$

$$\Rightarrow \underline{X}_n = \begin{bmatrix} -x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

So $\boxed{N(A^T) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$ ✓ ; $\boxed{\text{it is a line in } \mathbb{R}^3}$ ✓

Now do the same for U

Note the following : ① $R(A) = R(U)$
② $N(A) = N(U)$

hence $R(U)$ is already found, it is $R(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ ✓
and $N(U)$ " " " , " " $N(A) = \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ ✓

need to find $C(U)$ and Left Null (U).

$$\boxed{C(U) = \left[\left\{ \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \right\}, \left\{ \begin{matrix} 2 \\ 1 \\ 0 \end{matrix} \right\} \right]} \quad \text{since } U = \begin{bmatrix} \boxed{1} & 2 & 0 & 1 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} ✓$$

Now, $\text{Left Null}(U) = N(U^T) = N\left(\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right) = N\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)$

$$= N\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \Rightarrow \begin{array}{l} \text{pivot variables} = x_1 = 0, x_2 = 0 \\ \text{free variables} = x_3 \end{array} \Rightarrow N(U^T) = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

So $\boxed{\text{leftNull}(U) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$ ✓

Section 2.4 # 4

describe the 4 subspaces in $\mathbb{R}^3$ associated with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow C(A) = \left\{ \begin{matrix} 1 \\ 0 \\ 0 \end{matrix}, \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} \right\}$$

we see that A has 2 L.I. columns, so $\boxed{C(A) \text{ is a plane}}$ in $\mathbb{R}^3$. i.e $\boxed{DIM(C(A)) = 2}$

$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ pivot variables $= x_2, x_3$, free variables $= x_1$. $\Rightarrow$ $x_2 = 0$, $x_3 = 0$ $\Rightarrow \underline{X}_n = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

so $\boxed{N(A) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$. so $\boxed{N(A) \text{ is a Line in } \mathbb{R}^3}$ i.e $DIM(N(A)) = 1$

$R(A) = C(A') =$ Column space of $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow R(A) = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

so $\boxed{R(A) = \text{plane in } \mathbb{R}^3}$ i.e $DIM(Row(A)) = 2$

Left Null (A) is $N(A^T)$ since $A^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow$ free variables x_3 // pivot variables x_1, x_2 $\Rightarrow N(A^T) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x_3$

$\Rightarrow$

so $\boxed{N(A^T) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$ $\boxed{\text{a line in } \mathbb{R}^3}$

section 2.4 # 6

Suppose A is $m \times n$ matrix of rank r. Under what conditions on those numbers does

(a) A have a two-sided Inverse $AA^{-1} = A^{-1}A = I$

(b) $Ax = b$ have ∞ many solutions for every b?

(a) For a matrix to have a 2 sided inverse, it must be square matrix. hence $\boxed{n = m}$, and since it is Invertible, then it must be Full rank.

hence $\boxed{n = m = r}$ ✓

(b) Since ∞ many solutions, then we have something like

number of equations is $\underline{m}$

number of independent variables is $\underline{n}$

dimension of space solution exist in.

So we need as a condition for ∞ solution is to have the numbers of equations be <u>less than</u> the Dimension of the space of the solution. because if we had, in this example of using $\mathbb{R}^3$, 3 equations (i.e 3 planes), then we could possibly have these intersect at a <u>point</u>. o.k.

~~what if m>n but several equations (rows) were the same?~~

Therefor we know that $r < n$ is a condition for having ∞ solutions. however, we need to have these m equations be independent, so $\boxed{r = m}$ also. ✓

so $\boxed{r = m < n}$

Section 2.4 # 14

Find Left-inverse and/or right inverse when they exist for

$$A=\begin{bmatrix}1&1&0\\0&1&1\end{bmatrix},\quad M=\begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix},\quad T=\begin{bmatrix}a&b\\0&a\end{bmatrix}.$$

Answer For A: Left inverse $B \Rightarrow BA=I.$, right inverse $\Rightarrow AB=I.$

Since $\text{Rank}(A)=2$, which equalls m (number of rows), then only right inverse exist.

let $B=$

$$\underset{2\times3}{\begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}}\underset{3\times2}{\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\\a_{31}&a_{32}\end{bmatrix}}=\underset{2\times2}{\begin{bmatrix}1&0\\0&1\end{bmatrix}} \Rightarrow \left.\begin{matrix}a_{11}+a_{21}=1\\a_{12}+a_{22}=0\\a_{21}+a_{31}=0\\a_{22}+a_{32}=1\end{matrix}\right\}\text{ I can solve for these.}$$

But easier to use Formulas in Book, page 108:

$$\text{right Inverse}=A^T(AA^T)^{-1}=\begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix}\left(\underset{2\times3}{\begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}}\underset{3\times2}{\begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix}}\right)^{-1}$$

$$=\begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix}\left(\begin{bmatrix}2&1\\1&2\end{bmatrix}\right)^{-1}=\underset{3\times2}{\begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix}}\underset{2\times2}{\begin{bmatrix}2&1\\1&2\end{bmatrix}}^{-1}=\begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix}\left(\frac{1}{4-1}\begin{pmatrix}2&-1\\-1&2\end{pmatrix}\right)$$

$$=\boxed{\frac{1}{3}\begin{bmatrix}2&-1\\1&1\\-1&2\end{bmatrix}}$$

For M $m=3, n=2$, $\text{rank}=2 \Rightarrow \text{rank}=n$ Hence Left inverse $MB=I$

use Formula at page 108.

$$B=(M^TM)^{-1}M^T=\left[\underset{2\times3}{\begin{pmatrix}1&1&0\\0&1&1\end{pmatrix}}\underset{3\times2}{\begin{pmatrix}1&0\\1&1\\0&1\end{pmatrix}}\right]^{-1}\begin{pmatrix}1&1&0\\0&1&1\end{pmatrix}=\begin{bmatrix}2&1\\1&2\end{bmatrix}^{-1}\begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}=\frac{1}{3}\begin{bmatrix}2&-1\\-1&2\end{bmatrix}\begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}$$

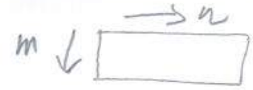
$$=\boxed{\frac{1}{3}\begin{bmatrix}2&1&-1\\-1&1&2\end{bmatrix}}$$

section 2.4 #14

$T = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ This is a square Matrix.

$T^{-1} = \frac{1}{a^2}\begin{bmatrix} a & -b \\ 0 & a \end{bmatrix}$ by using standard Formula For Matrix inverse on page 46.

So inverse exist if $a \neq 0$. and left inverse = right inverse.



section 2.4 # 15

IF the Columns of A are L.I. (A is $m \times n$), then the rank $\underline{\ n\ }$, the null space is $\underline{\text{contains only } \underline{0}}$, the row space is $\underline{\ R^n\ }$ and There exist a $\underline{\text{Left}}$ inverse.

Note since Full coulum rank, then rank $= n$. it can not be m. since rank is the smaller of n or m, and since n are L.I. it must be $= r$.

since $n = r$, then we have No free variables.
so Null(A) is empty (other than $\underline{0}$).

Section 2.4 #25

which Subspaces are the same for

(a) $[A]$ and $\begin{bmatrix} A \\ A \end{bmatrix}$ (b) $\begin{bmatrix} A \\ A \end{bmatrix}$, $\begin{bmatrix} A & A \\ A & A \end{bmatrix}$.

(Prove all matrices have same rank at end)

(a) Let $A_{m\times n}$. let rank $= r$. Then we have the following:

$C(A)$ has dimension $\underline{r}$ but subspace of R^m
$C(\begin{bmatrix} A \\ A \end{bmatrix})$ has dimension $\underline{r}$ but subspace of R^{2m}
} so $C(A)$ and $C(\begin{bmatrix} A \\ A \end{bmatrix})$ Not same.

$N(A)$ has dimension $n-r$, subspace of R^n
$N(\begin{bmatrix} A \\ A \end{bmatrix})$ has dimension $n-r$, subspace of R^n since n did not change.

so $N(A) = N\begin{bmatrix} A \\ A \end{bmatrix}$. same Null space

Row Space(A) has dimension $\underline{r}$, subspace of R^n
Row space $\begin{bmatrix} A \\ A \end{bmatrix}$ has dimension $\underline{r}$, subspace of $\mathbb{R}^n$
} Same Row space

Left Null (A), dimension $m-r$, Subspace of R^m
left Null $\begin{bmatrix} A \\ A \end{bmatrix}$, dimension $2m-r$, Subspace of R^{2m}
} Not same Left Null space

$\longrightarrow$

Section 2.4 # 25

(b) $\begin{bmatrix} A \\ A \end{bmatrix}$, $\begin{bmatrix} A & A \\ A & A \end{bmatrix}$.

$$\left.\begin{array}{l} C\left(\begin{bmatrix} A \\ A \end{bmatrix}\right) \text{ has dimension } \underline{r}, \text{ subspace of } \mathbb{R}^{2m} \\ C\begin{bmatrix} A & A \\ A & A \end{bmatrix} \text{ has dimension } \underline{r}, \text{ subspace of } \mathbb{R}^{2m} \end{array}\right\} \begin{array}{l}\text{hence same} \\ \underline{\text{Column space}}\end{array}$$

$$\left.\begin{array}{l} N\begin{bmatrix} A \\ A \end{bmatrix} \text{ has dimension } n-r, \text{ subspace of } \mathbb{R}^{n} \\ N\begin{bmatrix} A & A \\ A & A \end{bmatrix} \text{ has dimension } 2n-r, \text{ subspace of } \mathbb{R}^{2n} \end{array}\right\} \begin{array}{l}\text{Not same} \\ \underline{\text{Nullspace}}\end{array}$$

$$\left.\begin{array}{l} R\begin{bmatrix} A \\ A \end{bmatrix} \text{ has dimension } \underline{r}, \text{ subspace of } \mathbb{R}^{n} \\ R\begin{bmatrix} A & A \\ A & A \end{bmatrix} \text{ has dimension } \underline{r}, \text{ subspace of } \mathbb{R}^{2n} \end{array}\right\} \begin{array}{l}\underline{\text{Not same}} \\ \underline{\text{row space}}\end{array}$$

$$\left.\begin{array}{l} \text{Left Null}\begin{bmatrix} A \\ A \end{bmatrix} \text{ has dimension } 2m-r, \text{ subspace of } \mathbb{R}^{2m} \\ \text{Left Null}\begin{bmatrix} A & A \\ A & A \end{bmatrix} \text{ has dimension } 2m-r, \text{ subspace of } \mathbb{R}^{2m} \end{array}\right\} \begin{array}{l}\text{same} \\ \underline{\text{Left Null space}}\end{array}$$

<u>Note</u>

<u>some observation.</u>

if $C(A)$ is different, then Left Null is different.

If Row space is different, then Null space is different.

$\boxed{C(A)}$ $\boxed{\text{Left Null}}$

$\boxed{R(A)}$ $\boxed{N(A)}$

section 2.4 # 25 Proof of rank same:

rank is the dimension of the Column space of a matrix. it is the same as number of pivots. now, when we stack $\begin{bmatrix} A \\ A \end{bmatrix}$ to make one new matrix, then looking at the rows of this new matrix, we see that we have <u>not</u> added any new basis vectors. it is the same rows, just duplicated. so these new rows <u>do not</u> contribute any thing to increasing the dimension of the space spanned by the rows of ~~this~~ new matrix. hence the dimension of the row space did not change. but this is excatly what r is. it is also the dimension of the row space of A. so we see that rank of $[A]$ is same as rank $\begin{bmatrix} A \\ A \end{bmatrix}$.

now for the case of $\begin{bmatrix} A & A \\ A & A \end{bmatrix}$.

this is like writing $[C|C]$ where $C = \begin{bmatrix} A \\ A \end{bmatrix}$ which we allready showed it has rank r.

so now we have add a duplicate number of Columns. C has n columns, and we duplicated these. using same argument as befor, we say that These new Columns do <u>NOT</u> contribute anything new to the extent of the Column space that can be spanned allready. hence dim(Col space) did NOT change. $\Rightarrow$ rank same.

Section 2.4 #37

True or False

(a) A and A^T has same number of pivots.

$\boxed{\text{True}}$. rank is the number of L.I columns. it is also the number of L.I. rows. so transposing a matrix does not change its rank.

(b) A and A^T have same Left Null space.

Left Null Space has dimension of $\overset{\#\text{ of rows}}{m} - r$

rank is the same, but For Non square matrix, number of rows changes when transposing a matrix. Consider a row matrix of $1 \times n$, this has one row, but A^T has n rows.

so $\boxed{\text{FALSE}}$

(c) if row space = column space then $A^T = A$. $\Rightarrow$ $\boxed{\text{FALSE}}$

counter example

Let row space = column space, and let matrix be fully rank. i.e number of rows $\Rightarrow m$ = number of columns $\Rightarrow n$, and rank is the same as the size of this square matrix. since rank $= n$, and matrix is square, then we need to find an example of such matrix where $A^T \neq A$, i.e not symmetrical matrix.

here is one $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ this matrix has $C(A)$ $\mathbb{R}^2$. and has $R(A)$ $\mathbb{R}^2$.

but $A^T \neq A$.

Section 2.4 #37

(d) if $A^T = -A$, then Row space = Column space.

let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. then we have $\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} = -\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

i.e $a_{11} = -a_{11} \longrightarrow$ implies $a_{11} = 0$.

$a_{21} = -a_{12}$

$a_{12} = -a_{21}$

$a_{22} = -a_{22} \longrightarrow$ implies $a_{22} = 0$.

So A has <u>the Form</u> $\begin{bmatrix} 0 & c \\ -c & 0 \end{bmatrix}$

we see now that $C(A) = \left\{ \begin{bmatrix} c \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -c \end{bmatrix} \right\}$ i.e $\mathbb{R}^2$

and $R(A) = C(A^T) = \left\{ \begin{Bmatrix} 0 \\ c \end{Bmatrix}, \begin{Bmatrix} -c \\ 0 \end{Bmatrix} \right\}$ i.e $\mathbb{R}^2$

So $\boxed{\text{True}}$ row space = column space