Solution Key Math 307

Section 2.3 # 1, 5, 11, 19, 20, 34, 38

#1

$$c_1\begin{bmatrix}1\\0\\0\end{bmatrix}+c_2\begin{bmatrix}1\\1\\0\end{bmatrix}+c_3\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}}_{\vec{c}}=\underbrace{\begin{bmatrix}0\\0\\0\end{bmatrix}}_{\vec{0}}$$

Since A is invertible, there is a unique solution $\vec{c}=\vec{0}$. Hence the three vectors are independent.

if we add $\begin{bmatrix}2\\3\\4\end{bmatrix} \Rightarrow \begin{bmatrix}1&1&1&2\\0&1&1&3\\0&0&1&4\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\\c_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

Has c_4 as a free variable and therefore an infinite # of solutions, in particular $c_4=1$, so

$c_3+4=0 \Rightarrow c_3=-4$
$c_2+c_3+3(1)=0 \Rightarrow c_2=1$
$c_1+c_2+c_3+2(1)=0 \Rightarrow c_1=1.$

They are linearly dependent.

Note that if the matrix with the vectors as columns has an infinite # of solutions, then the columns are dependent, for a unique solution, they are independent.

#5 Decide the dependence or independence of

a) $\begin{bmatrix}1&2&3\\3&1&2\\2&3&1\end{bmatrix}\sim\begin{bmatrix}1&2&3\\0&-5&-7\\0&-1&-5\end{bmatrix}\sim\begin{bmatrix}1&2&3\\0&-1&-5\\0&-5&-7\end{bmatrix}\sim\begin{bmatrix}1&2&3\\0&-1&-5\\0&0&18\end{bmatrix} \Rightarrow$ invertible; i.e. independent.

b) $\begin{bmatrix}1&2&-3\\-3&1&2\\2&-3&1\end{bmatrix}\sim\begin{bmatrix}1&2&-3\\0&7&-7\\0&-7&7\end{bmatrix}\sim\begin{bmatrix}1&2&-3\\0&7&-7\\0&0&0\end{bmatrix} \Rightarrow$ infinite number of solutions, dependent.

#11

a) $\begin{bmatrix}1&-1\\1&-1\\-1&1\end{bmatrix}\sim\begin{bmatrix}1&-1\\0&0\\0&0\end{bmatrix}$ Linearly dependent $\Rightarrow$ a line. ; betteryet, $\begin{bmatrix}1\\1\\-1\end{bmatrix}=-1\begin{bmatrix}-1\\-1\\1\end{bmatrix}$ so it is the <u>line</u> spanned by either of the two vectors.

b) Really the easier thing to do is to treat them as rows, since elementary row operations are linear combinations of the rows.

$\begin{bmatrix}1&1&0\\0&1&1\\0&0&0\end{bmatrix} \rightarrow$ Two pivots, two linearly independent rows, we have a <u>plane</u> in $\mathbb{R}^3$.

We can also do it with the columns since $\begin{bmatrix}0\\0\\0\end{bmatrix}$ is dependent to all vectors and $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}$ are linearly independent.

#11 c) 2 pivots $\Rightarrow$ 2 linearly independent columns $\Rightarrow$ a plane.

d) All vectors with positive components. We could make an argument based on

$$S = \left\{ \vec{x} \in \mathbb{R}^3 : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ with } x_1, x_2, x_3 > 0 \right\}$$

But I think it is easier to work with a subset and show that it spans $\mathbb{R}^3$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow 3 \text{ pivots, hence } \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} \in S$$

and span $\mathbb{R}^3$.

#19 If $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent, the space they span has dimension n. (Since if they are linearly independent and they span the space, then they are a basis). If the vectors are columns of an m by n matrix, then m is <u>greater than or equal to</u> n because the number of pivots is n and there must be at least that many columns.

#20 a) $\begin{bmatrix} x_1 \\ x_1 \\ x_1 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

b) $x_1 + x_2 + x_3 + x_4 = 0 \Rightarrow x_1 = -x_2 - x_3 - x_4$

$$\begin{bmatrix} -x_2 - x_3 - x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

c) $\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$ and $\begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$ so $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $(*)$

We solve the linear system $(*)$ $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_3 = t$, $x_4 = v$, $-x_2 + x_3 + x_4 = 0$
$x_2 = t + v$

, $x_1 + x_2 = 0 \Rightarrow x_1 = -t - v$

so
$$\begin{bmatrix} -t-v \\ t+v \\ t \\ v \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

#20 a) The column space and row space of $\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$

column space in $\mathbb{R}^2$ has at most two basis vectors, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The row space in $\mathbb{R}^5$: $(1\,0\,1\,0\,1)$, $(0\,1\,0\,1\,0)$ since they are linearly independent

#34 If $\mathbb{V}$ is 3-dimensional, then every element $\vec{v} \in \mathbb{V}$ can be written as a linear combination of three basis vectors:

$$c_1 \vec{V}_1 + c_2 \vec{V}_2 + c_3 \vec{V}_3 = \vec{v}$$

also for $\mathbb{W}$:

$$c_4 \vec{W}_1 + c_5 \vec{W}_2 + c_6 \vec{W}_3 = \vec{W}$$

Now, we follow the hint and put them all toguether, and set them $= 0$.

$$c_1 \vec{V}_1 + c_2 \vec{V}_2 + c_3 \vec{V}_3 + c_4 \vec{W}_1 + c_5 \vec{W}_2 + c_6 \vec{W}_3 = 0$$

We have six vectors in a 5 dimensional space, they <u>must</u> be linearly dependent so there <u>must</u> be a solution to this equation for which <u>not all the constants are zero.</u>

Now we move them to opposite sides of the equation

$$c_1 \vec{V}_1 + c_2 \vec{V}_2 + c_3 \vec{V}_3 = -c_4 \vec{W}_1 - c_5 \vec{W}_2 + c_6 \vec{W}_3$$

For the same $\{c_n\}$ that were non-zero in the previous equation, we have that an element $\vec{v} \in \mathbb{V}$ is equal to an element $\vec{w} \in \mathbb{W}$, hence they must share a non-zero vector.

#38 The key to this one is to think of functions as vectors, then it is simple.

a) $\frac{dy}{dx} = 0 \Rightarrow y = c$ [solutions to homogeneous equation, equivalently the nullspace.]

b) $\frac{dy}{dx} = 3 \Rightarrow y = 3x$ [One particular solution that matches the right hand side.]

c) $\frac{dy}{dx} = 3$ has solutions $y = 3x + c$ [sum of particular solution with those in the nullspace!]

You see, you've been doing linear algebra for a long time...

Section 2.4 # 2, 4, 6, 14, 15, 25, 37

#2 Find the dimension and construct a basis for the four subspaces of:

$A = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix}$

column space $4\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} \Rightarrow$ Spanned by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, 1-dimensional.

row space $2 \cdot [0\ 1\ 4\ 0] = [0\ 2\ 8\ 0]$, so

rowspace is spanned by $[0\ 1\ 4\ 0]$, 1 dimensional.

Nullspace $\begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \overset{A_{21}(2)}{\sim} \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_2 + 4x_3 = 0$

$x_2 = -4x_3$

$\begin{bmatrix} x_1 \\ -4x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

3 Dimensional.

Left Nullspace

$[x_1\ x_2]\begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix} = [0\ 0\ 0\ 0]$

$\Rightarrow \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 4 & 8 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 + 2x_2 = 0$

$x_1 = -2x_2$

$\begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

1 dimensional.

$U = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ We can go through the same process to find that the row space and the nullspace are the same (i.e. are not changed by elementary row operations).

The Column Space is 1 dimensional, spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

The Left Nullspace is 1-dimensional, spanned by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

#4

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The column space is spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ i.e. x-y plane.

The row space is spanned by $[0\ 1\ 0], [0\ 0\ 1]$, i.e. The y-z plane.

The nullspace spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, x-axis

The left nullspace spanned by $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, z-axis

#6 A is an $m \times n$ matrix with rank r.

a) A has a two sided inverse $\Rightarrow$ A has to be square $m = n$
A has to be invertible $m = n = r$.

b) $A\vec{x} = \vec{b}$ have infinetely many solutions for every $\vec{b}$?

A has to have a right inverse (for solutions to exist)
$r = m$

There have to be an infinite # of solutions
$r < n$

#14 Find a left-inverse and/or right-inverse (when they exist) for

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$r = 2,\ m = 2,\ n = 3$

$r = m \Rightarrow$ right inverse exists.
$r < n \Rightarrow$ no left inverse.

Using equation for right-inverse (in section): $C = A^T (AA^T)^{-1}$

$$A^T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \quad AA^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad (AA^T)^{-1} = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}$$

$$A^T (AA^T)^{-1} = \begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix}$$

#14 continued

$M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ $r=2,\ m=3,\ n=2$

$r=n,\ r<m \Rightarrow$ Left inverse exists.

$B = (A^TA)^{-1}A^T = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$

$T = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ if $a \neq 0$, T has a 2-sided inverse $\begin{bmatrix} 1/a & -b/a^2 \\ 0 & 1/a \end{bmatrix}$

if $a=0$, T has rank $1 < m=n=2$, no left or right inverse!

#15 If the columns of A are linearly independent (A is m by n), then the rank is $\underline{n}$ (since number of pivots = # of columns), the nullspace is <u>only the zero vector</u> since $A\vec{x}=\vec{0}$ has a unique solution (columns are linearly independent). The row space is n-dimensional (dimension is the same as the column space) and hence is $\mathbb{R}^n$. Since $r=n$, there is a <u>left</u> inverse, because you need as many pivots as columns for $A^T\vec{y}=b$ to always have a solution.

#25 $[A]$ has rank r

$\begin{bmatrix} A \\ A \end{bmatrix}$ has rank equal to the dimension of the row space $=r$

$\begin{bmatrix} A & A \\ A & A \end{bmatrix}$ has rank equal to the dimension of the column space which is the same space as the column space of $\begin{bmatrix} A \\ A \end{bmatrix}$ which has the same dimension as it's rowspace

So, in "math" $\dim\left[C\left(\begin{bmatrix} A & A \\ A & A \end{bmatrix}\right)\right] = \dim\left[C\left(\begin{bmatrix} A \\ A \end{bmatrix}\right)\right] = \dim\left[C\left([A\ A]\right)\right] = r.$

a) $[A]$ and $\begin{bmatrix} A \\ A \end{bmatrix}$ have the rowspace since the rows are the same, and the nullspace since

$A\vec{x}=0$ and $\begin{bmatrix} A \\ A \end{bmatrix}\begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ have the same solutions

The column space and left-nullspace are of different sizes.

b) $\begin{bmatrix} A \\ A \end{bmatrix}$ and $\begin{bmatrix} A & A \\ A & A \end{bmatrix}$ has the column space and left nullspace since

$[A\ A]\vec{y} = \vec{0}$ and $\begin{bmatrix} A & A \\ A & A \end{bmatrix}\vec{y} = \vec{0}$ have the same solutions.

The rowspace and nullspace are of different sizes.

#37 True or False

a) A and A^T have the same number of pivots because the dimension of the rowspace and columnspace is the same, True. (same rank).

b) A and A^T dont have the same left nullspace, let $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $A^T = [1\ 0]$, the left nullspaces are of different sizes!, False

c) The row space of any square invertible matrix is equal to the column space but it does not have to be symmetric. Counter example $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. False.

d) If $A^T = -A$, then the rows of $-A$ are the column space of A^T, hence they are linear combinations of each other. True.