19/20 nice!

HW4, Math 307. CSUF. Spring 2007.

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2.1 # 2, 3, 5, 8, 9, 24, 28

2.2 # 1, 2, 7, 30, 39, 45, 50

Please Note: Section 2.3 problem moved to HW #5 as OK as mentioned in class. will be handed as part of HW #5 due next wed.

HW took ~ 12 hrs (including time to study for it)

Contents

1 Section 2.1, problem 2

Problem: Which of the following subsets of R^3 are actually subsets?

(a) The plane of vectors (b_1, b_2, b_3) with the first component $b_1 = 0$

(b)The plane of vectors b with $b_1 = 1$

(c)The plane of vectors b with $b_2 b_3 = 0$ (this is the union of the 2 subspaces, plane $b_2 = 0$ and plane $b_3 = 0$)

(d)All combinations of 2 vectors $(1, 1, 0)$ and $(2, 0, 1)$

(e)The plane of vectors (b_1, b_2, b_3) which satisfies $b_3 - b_2 + 3b_1 = 0$

-1

answer:

(a) This is the plane $y - z$ which $x = 0$. The zero vector is in this plane. Adding any 2 vectors will remain this this plane, and scaling any vector will remain in the plane. Hence this is a $\boxed{\text{subspace of } R^3}$ ✓ True but I would like to see the algebraic argument.

(b) This is a plane parallel to the $y - z$ plane, but at an offset of 1. Hence this plane does not include the origin. Hence this is $\boxed{\text{not a subspace of } R^3}$ ✓ same here, you are correct but showing this plane is not closed under vector addition + scalar multiplication is important

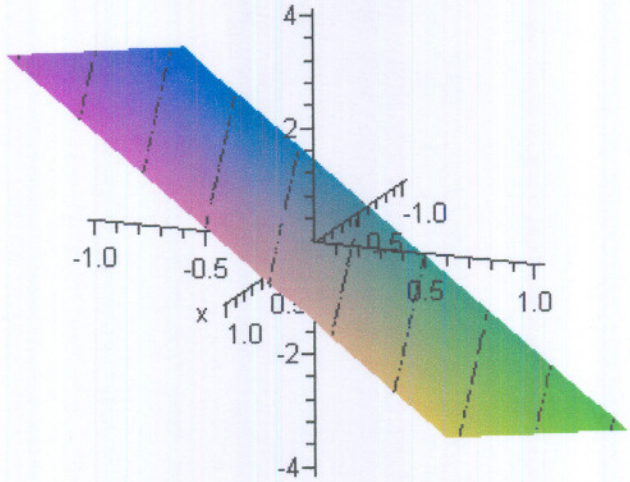
(c) This is the union of the plane $x - z$ and the plane $x - y$, hence this plane ~~is~~ the $x - axis$ only. The zero vector is one this line. Adding any 2 vectors will remain this this line, and scaling any vector will remain one the line. Hence this is a $\boxed{\text{subspace of } R^3}$ X do the algebra!

(d) Since these 2 vectors are linearly independent (There is no way to multiply one of these vectors by a scalar to obtain the second one) hence they are bases vectors for an R^2 space within R^3. Since the zero vector can be reached by these 2 vectors (they both originate from the origin), Hence this is a $\boxed{\text{subspace of } R^3}$ (by definition, bases vectors can be used to reach any vector in the space these bases vectors span), and since the problem did not give any restriction to the region within this subspace in which we can reach, hence closed under addition and scaler multiplication). ✓

(e)These vectors span a plane through the origin in which each vector satisfy the equation on its coordinates such that $b_3 - b_2 + 3b_1 = 0$. This an illustration on this plane ($z = y - 3x$)

$$\text{plot3d}(y - 3 * x, y = -1..1, x = -1..1)$$

Working through the algebra makes your arguments precise.



The zero vector is in this subspace. OK

Any scaled vector on this plane will remain in this plane. OK.

The additions of any 2 vectors in this plane will remain in this plane. (the plane is 'flat', i.e. it has zero curvature, hence it is not possible to leave this plane using combination of vectors which lie in the plane itself).

Hence this is a $\boxed{\text{subspace of } R^3}$ ✓

2 Section 2.1, problem 3

question:

Describe the column space and the null space of these matrices

$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 2 & 3 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

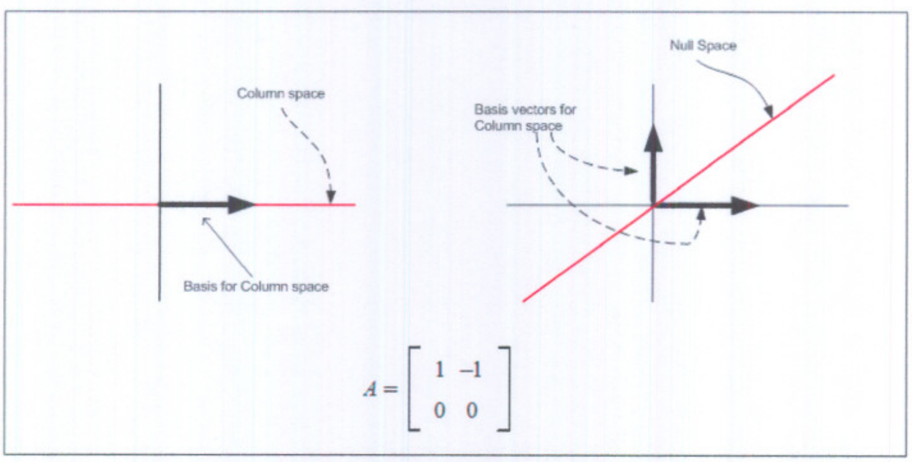
answer:

A)

The column space is the space reached by vectors represented by the columns of the matrix. For matrix A these are the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, hence the space is the set of all vectors $a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} -1 \\ 0 \end{pmatrix}$, for $a, b \in R$, hence this is the $\boxed{\text{whole } x - axis \text{ line.}}$ i.e. $\boxed{C(A) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$ i.e. the vector $(1, 0)$ is the basis for the column space of A.

The null space is the solution to $Ax = 0$, hence we have $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which gives $x - y = 0$ which means $x = 1$ and $y = 1$ (or $x = c$, $y = c$ for any scalar c). i.e. $\boxed{N(A) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$ i.e. the vector $(1, 1)$ is the basis vector for the Null space of A.

Hence the null space is a line whose coordinates are (c, c)



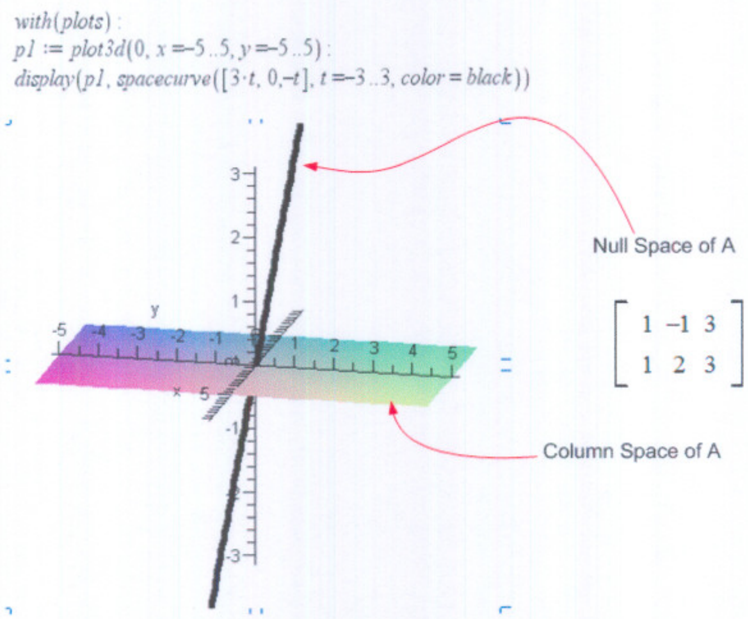
$B = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

The first and third columns do not produce any new basis, hence the column space is spanned by the first and the second columns only. These 2 columns are linearly independent, hence they are basis. Since we have 2 basis vectors, and we in are in R^3 space, hence these 2 columns will span all of the subspace R^2 and a basis that can be used to span the column space are the the standard basis $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

The null space is the solution to $Ax = 0$, or $\begin{bmatrix} 1 & -1 & 3 \\ 1 & 2 & 3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, hence we obtain

$x_1 - x_2 + 3x_3 = 0$, and $x_1 + 2x_2 + 3x_3 = 0$

If we try $x_1 = 3, x_2 = 0, x_3 = -1$, we see that this solution satisfies these 2 equations. Hence $\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$ is a basis vector that spans the null space of A. The null space of A is a line through the origin. Any point on the line $3x + 0y - z = 0$ is in the null space of A



$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The column space is the space that can be reached by combination of the columns. Since the columns are the zero vectors, hence the space that can be spanned is empty. Hence $\boxed{C(A) = \{\}}$

Since the dimension of the space reached by the column space basis + the dimension of the space reach by the null space basis must equal the number of columns which is 3, hence the $\boxed{\text{null space must be } R^3}$, and the basis that can be used are the standard basis

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

3 Section 2.1 problem ~~6~~ 5

(a) Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

note: in these problems use the zero vector as $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$

Rules (1),(2) can easily be verified to be valid under the new sum rule.

For rule (3), we get

$$\begin{aligned}
\mathbf{x} &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \mathbf{0} \\
&= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} x_1 - 1 + \boxed{1} \\ x_2 - 1 + \boxed{1} \end{pmatrix} \\
&= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= \mathbf{x}
\end{aligned}$$

hence rule(3) is NOT broken.

For rule (4) we have

$$\begin{aligned}
\mathbf{0} &= \mathbf{x} + (-\mathbf{x}) \\
&= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix} \\
&= \begin{pmatrix} x_1 - x_1 - \boxed{1} \\ x_2 - x_2 - \boxed{1} \end{pmatrix} \\
&= \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \mathbf{0} \quad \text{(using these rules)}
\end{aligned}$$

hence rule (4) is not broken.

Rules (5) and (6) are not affected since they involve only multiplication.

For rule (7) we have

$$
\begin{aligned}
c\,(\mathbf{x}+\mathbf{y}) &= c\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) \\
&= c\begin{pmatrix} x_1 + y_1 + \boxed{1} \\ x_2 + y_2 + \boxed{1} \end{pmatrix} \\
&= \begin{pmatrix} cx_1 + cy_1 + c \\ cx_2 + cy_2 + c \end{pmatrix} \\
&= \begin{pmatrix} cx_1 \\ cx_2 \end{pmatrix} + \begin{pmatrix} cy_1 \\ cy_2 \end{pmatrix} + \begin{pmatrix} c \\ c \end{pmatrix} \\
&= c\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + c\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + c\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= c\mathbf{x} + c\mathbf{y} + c\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= c\,(\mathbf{x}+\mathbf{y}) + c\begin{pmatrix} 1 \\ 1 \end{pmatrix}
\end{aligned}
$$

but $c\,(\mathbf{x}+\mathbf{y}) \neq c\,(\mathbf{x}+\mathbf{y}) + c\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, hence rule (7) is broken

For rule 8,

$$
\begin{aligned}
(c_1+c_2)\,\mathbf{x} &= (c_1+c_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= c_1\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + c_2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= \begin{pmatrix} c_1x_1 \\ c_1x_2 \end{pmatrix} + \begin{pmatrix} c_2x_1 \\ c_2x_2 \end{pmatrix} \\
&= \begin{pmatrix} c_1x_1 + c_2x_1 + \boxed{1} \\ c_1x_2 + c_2x_2 + \boxed{1} \end{pmatrix} \\
&= \begin{pmatrix} c_1x_1 + c_2x_1 \\ c_1x_2 + c_2x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= c_1\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + c_2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= (c_1+c_2)\,\mathbf{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\end{aligned}
$$

but $(c_1+c_2)\,\mathbf{x} \neq (c_1+c_2)\,\mathbf{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, hence rule (8) is broken.

Hence rules $\boxed{\text{(3),(4),(7),(8) are broken}}$

section 2.1 # 8

which of the following descriptions is correct? solutions x of

$$Ax = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\uparrow \quad \uparrow$

from
(a) plane
(b) line
(c) point
(d) subspace
(e) $N(A)$
(f) $C(A)$

Solution

Since $A\underline{x} = \underline{0}$, here we are only looking at the homogeneous solution. (i.e. Null space).

Column space is a subset of R^2 → number of rows.

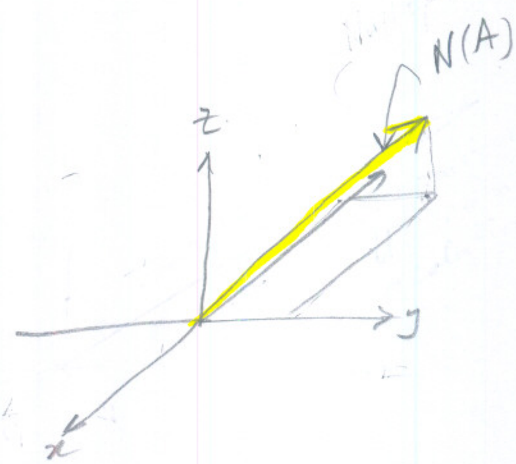
Null space is subset of R^3 → number of columns.

Reduce $A\underline{x} = \underline{0}$ (reduce Row echelon Form) ⟹

$$\begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{-1} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\underbrace{\uparrow_{c_1} \quad \uparrow_{c_2}}$

These 2 Columns mean That column 1 and Column 2 of A are basis for $C(A)$



To Find solution:

$-x_2 + x_3 = 0 \Rightarrow \boxed{x_3 = x_2}$

$x_1 + x_2 + x_3 = 0 \Rightarrow x_1 = -x_2 - x_3$.

so if $x_3 = t$, $x_2 = t$, $x_1 = -2t$

$\boxed{\underline{x} = \begin{bmatrix} -2t \\ t \\ t \end{bmatrix}}$ ← solution $= t\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

So solution $\boxed{\underline{x} \text{ lives in the Null space}}$ which is a Line in R^3.

the solutions $\bar{x}$ are a subspace also. (Nullspace contains origin, and closed under addition and scalar multiplication)

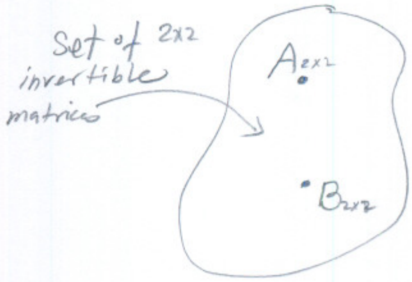
it is the Null space of A.

Hence $\boxed{(b),\ (d),\ (e)}$ are correct

section 2.1 # 9

Show that the set of nonsingular 2x2 matrices is Not a Vector space. Show also that the set of singular 2x2 matrices is Not a Vetor space.

Answer



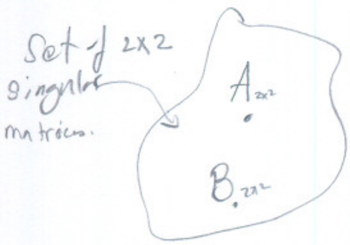
for this set to be Vector space, A "zero" Vector must also be in this set.

let $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}$.

then $A+(-A)=\begin{pmatrix} 0 & 0 \\ 0 & 0\end{pmatrix}$ the "zero" Vector

BUT $\begin{pmatrix} 0 & 0 \\ 0 & 0\end{pmatrix}$ is singular hence it is NOT in the set.

a set that does not include a "zero" Vector can NOT be a Vector space. hence $\boxed{\text{this set is NOT a vector space}}$



All what we have to show is 2 "vectors" in this set which add to a "vector" outside this set.

let $A=\begin{pmatrix} 0 & 0 \\ 1 & 2\end{pmatrix}$. this is clearly singular as it has a row of zeros

let $B=\begin{pmatrix} 2 & 1 \\ 0 & 0\end{pmatrix}$ this is also singular.

but $A+B=\begin{pmatrix} 0 & 0 \\ 1 & 2\end{pmatrix}+\begin{pmatrix} 2 & 1 \\ 0 & 0\end{pmatrix}=\begin{pmatrix} 2 & 1 \\ 1 & 2\end{pmatrix}$ but this is NOT singular. as its determinant is $3\neq 0$.

hence this set is NOT closed under addition. $\Rightarrow \boxed{\text{NOT a vector space}}$

section 2.1 # 24

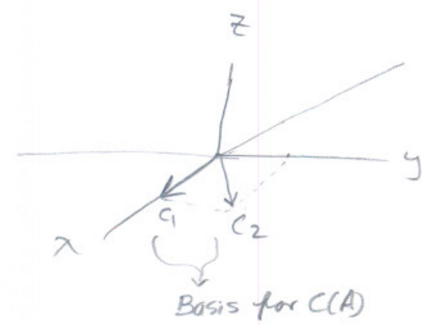
for which vector (b_1, b_2, b_3) do these systems have a solution?

$$\begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & \boxed{1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

we see A is full column Rank, hence $C(A)$ is the full R^3.

so $\boxed{\text{any vector } \underline{b}}$ in R^3 will give a solution $\underline{X}$.

$$\begin{array}{c} \begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & \textcircled{1} & 1 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccc} c_1 & c_2 & \end{array} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$



hence $C(A)$ is subset of R^3.

so $C(A)$ is the $\underline{\text{x-y plane}}$.

since the basis c_1, c_2 can span any point on the x-y plane by linear combination of the basis.

so only $\underline{b}$ vectors which lie in the x-y plane will give a solution. i.e. $\begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix}$

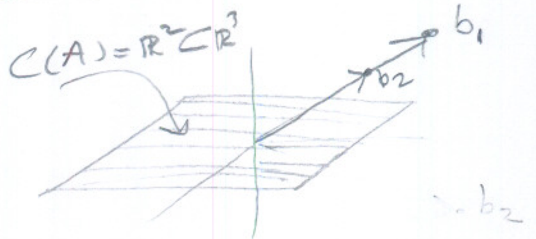
section 2.1 # 28

True or False.

(a) Vector $\underline{b}$ that are not in $C(A)$ form a subspace.

given $A_{m \times n}$, then $C(A)$ is subset of $\mathbb{R}^m$.

Vectors $\underline{b}$ not in $C(A)$ must ~~the~~ live in "subspace" of dimension $\boxed{m-r}$. This space includes the origin. so need to see if closed under addition and scalar multiplication. Imagin $m=3$. next assume $r=2$. then $C(A)$ is a <u>plane</u> in $\mathbb{R}^3$. so $\underline{b}$ that can <u>not</u> be reached must be a point "outside" this <u>plane</u>. imagine, we have 2 such points (i.e 2 such $\underline{b}$) both outside the "plane" which is $C(A)$. as follows



we see that $C(A)$ is a plane in $\mathbb{R}^3$.

now we ask; can any combination of b_1, b_2 lead to position vector which is NOT in the space of b_1, b_2 in which they are on now? The space that $\underline{b}$ is on is a line through the origin. we see that No matter what combination of Vectors on this line we "add", we could never leave the line. hence space occupied by $\underline{b}$ Not in $C(A)$ makes a subspace.

$$\boxed{\text{TRUE}}$$

section 2.1 #28

(b) if $C(A)$ Contains only zero vector, then A is zero Matrix.

$\boxed{\text{TRUE}}$. because the minimum number of Linearly independent Columns in a Matrix A is 1. (i.e one pivot). if we can't find at least one pivot in the matrix, then this means all entries are zero.

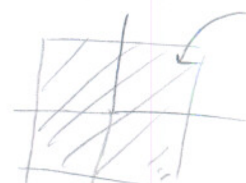
(c) the Column space of $2A$ equals the Column space of A.

$\boxed{\text{TRUE}}$ multiplication of a matrix by scalar does NOT change the Linear independence or the Linear dependence between Columns.

(d) the Column space of $A-I$ equals the Column space of A.

$\boxed{\text{FALSE}}$ Counter example.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ this has $C(A)$ as the complete x-y plane



$C(A)$.

$A-I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, this has $C(A)$ as the zero Vector only. $C(A) = \{\underline{0}\}$. (see (b)).

so $C(A-I) \neq C(A)$

Section 2.2 # 1

Construct system with more unknowns than equations, but No solution. change RHS to zero and find X_n

Solution

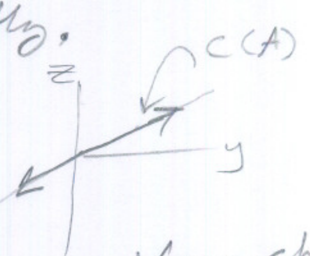
One way to construct such system is to consider 2 planes in 3D such that they are parallel to each others. hence No solution. the 2 equations are

$$z-x-y=b_1 \qquad\qquad \begin{pmatrix}-1 & -1 & 1\\ -1 & -1 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}b_1\\b_2\end{pmatrix}$$
$$z-x-y=b_2 \qquad \rightarrow$$

$$\xrightarrow{l_{21}=1} \begin{pmatrix}\boxed{-1} & -1 & 1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}b_1\\b_2-b_1\end{pmatrix}$$

we see that for system to be consistant then $\boxed{b_2-b_1=0}$ i.e $b_2=b_1$. i.e the 2 planes are on top of each others exactly.

$$C(A)=\left\{\begin{matrix}-1\\0\end{matrix}\right\}$$



so for solution to exist, $\underline{b}$ must be along the x-axis.

Now change RHS to zero

$$\begin{pmatrix}-1 & -1 & 1\\ -1 & -1 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\rightarrow\begin{pmatrix}\boxed{-1} & -1 & 1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

so pivot variable is x, free variables is y, z.

$$-x-y+z=0 \Rightarrow x=z-y \Rightarrow \underline{X}=\begin{pmatrix}z-y\\y\\z\end{pmatrix}$$

let $y=t_1$, $z=t_2$ we write $\boxed{\underline{X}=\begin{pmatrix}t_2-t_1\\t_1\\t_2\end{pmatrix}}$

to verify: let $t_2=5, t_1=1 \Rightarrow$ $y=1, z=5, x=5-1=4$.

so from first eq: $-x-y-z=0 \Rightarrow -(4)-(1)+(5)=0$ ok.

Section 2.2 #2

Reduce A and B to echelon form to find their ranks. Which variables are free?

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

Solution

$$A \xrightarrow{l_{31}=1} \begin{pmatrix} \boxed{1} & 2 & 0 & 1 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B \xrightarrow[l_{31}=7]{l_{21}=4} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix} \xrightarrow{l_{32}=2} \begin{pmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{-3} & -6 \\ 0 & 0 & 0 \end{pmatrix}$$

hence $\text{Rank}(A) = 2$

$\text{Rank}(B) = 2$

To find solutions

$\boxed{\text{For } A:}$

$$\begin{pmatrix} \boxed{1} & 2 & 0 & 1 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} \boxed{1} & 0 & -2 & 1 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix} \longrightarrow \boxed{\begin{array}{l} \text{Pivot variables: } u, v \\ \text{free variables: } w, y. \end{array}}$$

$$\left.\begin{array}{l} u - 2w + y = 0 \\ v + w = 0 \end{array}\right\} \longrightarrow \begin{array}{l} u = 2w - y \\ v = -w \end{array} \Rightarrow \underline{X} = \begin{pmatrix} 2w - y \\ -w \\ w \\ y \end{pmatrix} \text{ or } \underline{X} = \underbrace{\begin{pmatrix} 2t_1 - t_2 \\ -t_1 \\ t_1 \\ t_2 \end{pmatrix}}_{x_n}$$

For B:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \longrightarrow \boxed{\begin{array}{l} \text{Pivot variables: } u, v \\ \text{free variables: } w. \end{array}}$$

so
$$\left.\begin{array}{l} u - w = 0 \\ v + 2w = 0 \end{array}\right\} \longrightarrow \begin{array}{l} u = w \\ v = -2w \end{array} \longrightarrow \underline{X} = \begin{pmatrix} w \\ -2w \\ w \end{pmatrix} \text{ or } \underline{X} = \begin{pmatrix} t \\ -2t \\ t \end{pmatrix}$$

Section 2.2 # 7

Find the value of c that makes it possible to solve $Ax=b$ and solve it

$$\begin{aligned} u+v+2w &= 2 \\ 2u+3v-w &= 5 \\ 3u+4v+w &= c \end{aligned} \Rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & -1 \\ 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ c \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -5 \\ 0 & 1 & -5 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ c-6 \end{pmatrix} \rightarrow \boxed{\begin{pmatrix} \boxed{1} & 1 & 2 \\ 0 & \boxed{1} & -5 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ c-7 \end{pmatrix}} \quad (1)$$

here $C(A) = \left\{\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}\right\}, \left\{\begin{matrix} 1 \\ 3 \\ 4 \end{matrix}\right\}$ hence for solution to exist $\underline{b}$ must lie in the space spanned by $C(A)$.

So need to find some linear combination of $C(A)$ to give b.

$k_1 \left\{\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}\right\} + k_2 \left\{\begin{matrix} 1 \\ 3 \\ 4 \end{matrix}\right\} = \begin{pmatrix} 2 \\ 5 \\ c \end{pmatrix}$. we see that $k_1=1, k_2=1$ work $\Longrightarrow 3+4=c$

so $\boxed{c=7}$

now we solve by back substitution. but first generate 'R':

plug $c=7$ in (1)

$$\Rightarrow \begin{pmatrix} \boxed{1} & 1 & 2 \\ 0 & \boxed{1} & -5 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \boxed{u} \\ \boxed{v} \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{RREF} \boxed{\underset{R}{\begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{pmatrix}}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$$

u, v pivot variables

w free variable. $\equiv t$

$$\begin{aligned} u+7w&=1 \\ v-5w&=1 \end{aligned} \Rightarrow \left.\begin{aligned} u&=1-7w \\ v&=1+5w \end{aligned}\right\} \Rightarrow \underline{X} = w\begin{pmatrix} -7 \\ 5 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{or } \boxed{\underline{X} = \begin{pmatrix} -7t+1 \\ 5t+1 \\ 1 \end{pmatrix}}$$

where $w \equiv t$

so ∞ solutions depending on t value we get a solution

solution can also be written as

$$\boxed{\underline{X} = t\underbrace{\begin{pmatrix} -7 \\ 5 \\ 1 \end{pmatrix}}_{X_n} + \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}_{X_p}}$$

Section 2.2 #30

execute 6 Steps to find $C(A)$ and $N(A)$ and solution of

$$A=\begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix}, \quad b=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}=\begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

Solution

(1)
$$\left[\begin{array}{cccc|c} 2 & 4 & 6 & 4 & b_1 \\ 2 & 5 & 7 & 6 & b_2 \\ 2 & 3 & 5 & 2 & b_3 \end{array}\right] \to \left[\begin{array}{cccc|c} \boxed{2} & 4 & 6 & 4 & b_1 \\ 0 & \boxed{1} & 1 & 2 & b_2-b_1 \\ 0 & -1 & -1 & -2 & b_3-b_1 \end{array}\right] \to \left[\begin{array}{cccc|c} 2 & 4 & 6 & 4 & b_1 \\ 0 & 1 & 1 & 2 & b_2-b_1 \\ 0 & 0 & 0 & 0 & b_3-b_1-(b_1-b_2) \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} \boxed{2} & 4 & 6 & 4 & b_1 \\ 0 & \boxed{1} & 1 & 2 & b_2-b_1 \\ 0 & 0 & 0 & 0 & b_3-2b_1+b_2 \end{array}\right] \checkmark$$

$U \qquad C$

(2) Condition is $b_3-2b_1+b_2=0$ for Consistant system. ✓

(3) $C(A)$ is plane spanned by 2 vectors $\left\{\begin{matrix}2\\2\\2\end{matrix}\right\}, \left\{\begin{matrix}4\\5\\3\end{matrix}\right\}$. ✓

another description: $C(A)$ contains all $\underline{b}$ vectors with $b_3-2b_1+b_2=0$

(4) free variables: x_3, x_4. pivot Variables: x_1, x_2

$$A\underline{x}=\underline{0} \Rightarrow \begin{bmatrix} 2 & 4 & 6 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xRightarrow{\text{RREF}} \begin{bmatrix} 2 & 0 & 2 & -4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{matrix} 2x_1+2x_3-4x_4=0 \\ x_2+x_3+2x_4=0 \end{matrix}\right\} \Rightarrow \begin{matrix} x_1=2x_4-x_3 \\ x_2=-2x_4-x_3 \end{matrix} \Rightarrow \underline{x}_n=\begin{bmatrix} 2x_4-x_3 \\ -2x_4-x_3 \\ x_3 \\ x_4 \end{bmatrix}. \checkmark$$

$$x_3=1, x_4=0 \Rightarrow \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad x_3=0, x_4=1 \Rightarrow \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix} \Rightarrow N(A)=\underbrace{\begin{bmatrix} -1 & 2 \\ -1 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{special solution in } N(A)}$$

$\longrightarrow$

⑤ $Ax = \begin{pmatrix}4\\3\\5\end{pmatrix}$. From $Ux = c \Rightarrow$

$$\begin{bmatrix} 2 & 4 & 6 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3-4 \\ 5-2(4)+3 \end{bmatrix} \Rightarrow \overset{U}{\begin{bmatrix} \boxed{2} & 4 & 6 & 4 \\ 0 & \boxed{1} & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \overset{c}{\begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}}$$

pivot variables $= \{x_1, x_2\}$, free variables $= \{x_3, x_4\}$.

$$\Rightarrow \left.\begin{aligned} 2x_1 + 4x_2 + 6x_3 + 4x_4 &= 4 \\ x_2 + x_3 + 2x_4 &= -1 \end{aligned}\right\} \text{ with free variable set to } 0 \Rightarrow$$

$$\left.\begin{aligned} 2x_1 + 4x_2 &= 4 \\ x_2 &= -1 \end{aligned}\right\} \quad x_1 = \frac{4-4x_2}{2} = \frac{4+4}{2} = 4, \quad x_2 = -1$$

so $X_p = \begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix}\begin{matrix} \\ \\ \text{free} \\ \text{free} \end{matrix}$

Complete solution $\underline{X} = X_p + X_n = \boxed{\begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}}$

or $\underline{X} = \begin{bmatrix} 4 - x_3 + 2x_4 \\ -1 - x_3 - 2x_4 \\ x_3 \\ x_4 \end{bmatrix}$

⑥ Reduce $[U \quad c]$ to $[R \quad d]$:

$$\underset{[\;U \;|\; c\;]}{\left[\begin{array}{cccc|c} \boxed{2} & 4 & 6 & 4 & 4 \\ 0 & \boxed{1} & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]} \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 2 \\ 0 & \boxed{1} & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \underset{[\;R \;|\; d\;]}{\left[\begin{array}{cccc|c} \boxed{1} & 0 & 1 & -2 & 4 \\ 0 & \boxed{1} & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]}$$

So $[R|d] = \underset{[\;R \;|\; d\;]}{\left[\begin{array}{cccc|c} \boxed{1} & 0 & 1 & -2 & 4 \\ 0 & \boxed{1} & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]}$

2.2 #39

Explain why these statements are false:

(a) the complete solution is any linear combination of x_p and x_n.

Consider $Ax = \underline{b}$. here x is the complete solution.

now assume $x = c_1 x_p + c_2 x_n$ where c_1, c_2 are any constants.

Then we write

$$A x = \underline{b}$$

$$A(c_1 x_p + c_2 x_n) = \underline{b}$$

$$A(c_1 x_p) + A(c_2 x_n) = \underline{b}$$

$$c_1 A x_p + c_2 A x_n = \underline{b}.$$

but $\boxed{A x_n = \underline{0}}$ by definition.

So

$$c_1 A x_p = \underline{b}$$

i.e. $\boxed{A x_p = \frac{1}{c_1}\underline{b}}$ —— (1)

Therefore since also $A(x_p + x_n) = \underline{b}$, we then have

$$A x_p + A x_n = \underline{b}$$

$$A x_p + \underline{0} = \underline{b} \Rightarrow \boxed{A x_p = b} \text{ —— (2)}$$

Now compare (1) and (2) $\Rightarrow$

then $\boxed{\frac{1}{c_1}\underline{b} = \underline{b}}$

so we see this is NOT true unless $\boxed{c_1 = 1}$

so Not any linear combination will be acceptable. c_1 must be 1. While c_2 can be any other value.

Hence FALSE statement.

Section 2.2 # 39

(b) The system $Ax=b$ has at most one particular solution:

this is FALSE since there are ∞ number of particular solutions. we generated x_p by setting all free variables to zero in the $Ux=b$ equation. this gave us one particular solution x_p.

by $\boxed{\text{setting the free variables to another value}}$ we can obtain another x_p different from the one we obtained when we set the free variables to zero.

(c) the solution x_p with all free variables zero is the shortest solution.

let the U matrix be $\begin{bmatrix} \boxed{1} & 2 \\ 0 & 0 \end{bmatrix}$, and $c = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

so pivot variable $= x_1$, free variable $= x_2$.

so $x_1 + 2x_2 = 2 \Rightarrow \boxed{x_1 = 2 - 2x_2}$

now to find x_p, set free variable to zero. $\rightarrow \boxed{x_p = \begin{bmatrix} 2 \\ 0 \end{bmatrix}}$

now try different value for free variable. say $x_2 = 1$

$$\Rightarrow \quad x_1 = 2 - 2(1) = 0.$$

so $\quad x_p = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

so clearly $\boxed{x_p = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \text{ is NOT the minimum}}$, even though we selected zero for free variable.

Complete Solution with $x_2 = 0$

$$X = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Complete. Solution with $x_2 = 2$ $\quad X = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

section 2.2 # 39

(d) if A is invertible then is no solution in Nullspace.

$x = \underline{0}$ is a solution for $A\underline{x} = \underline{0}$ which allows A to be invertible, since 'OK' to get $\underline{0}$ back to $\underline{0}$.

but $\underline{0}$ is in the Null space.

So statment is FALSE. i.e. solution can be in $N(A)$ and yet A be invertible.

Section 2.2 # 45

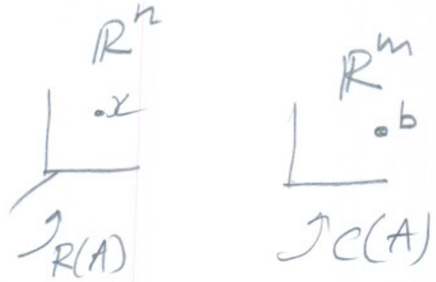
Write down all relations between r and m and n if $Ax=b$

When (a) No solution for some b

(b) ∞ many solutions for every b

(c) exactly one solution for some b, No solution for other b

(d) exactly one solution for every b.



Answer

(a)

$$m \downarrow \overset{\longrightarrow n}{\boxed{\quad A \quad}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_m \end{bmatrix} \longrightarrow$$

Row space is $\mathbb{R}^n$.

Column space is $\mathbb{R}^m$ or less than $\mathbb{R}^m$. (i.e subset of $\mathbb{R}^m$). if A has Full Column rank, then $r=m$.

$\underline{b}$ is a point that exist in $\mathbb{R}^m$.

Therfor if we can "reach" every $\underline{b}$, then this means that $C(A) = \mathbb{R}^m$. i.e $C(A)$ has full Column Rank. i.e $r=m$. but we are told that some $\underline{b}$ can NOT be reached and some can. therfor

$\boxed{r<m}$

because $C(A)$ is a subset of $\mathbb{R}^m$ in this case. and r tells us the # of Linear Independent Columns, or the Column rank.

Now for r and n relation.

if $C(A)$ can reach every $\underline{b}$ in $\mathbb{R}^m$, then $r=m$. and if $n=m$, then in this case only $\boxed{r=n}$.

but if as the case here, not all $\underline{b}$ can be reached, then $r<m<n$ i.e $\boxed{r<n}$.

book gives $\boxed{r \le n}$. I do not understand it, I think answer should be $\boxed{r<n}$ only, since NOT all $\underline{b}$'s are reached. I think book wrong.

(1)

Section 2.2 # 45

(b) infinitely many solutions for every $\underline{b}$.

Since every $\underline{b}$ can be reached, then we know right away that $\boxed{r = m}$ because $\underline{b}$ lives in $\mathbb{R}^m$ and so m Linearly independent vectors are need to span all of $\mathbb{R}^m$. so column Rank of $A = m$.

but since there are ∞ ways to combine these vectors to obtain $\underline{b}$, then A must have <u>more Columns</u> than just r. i.e fat A.

i.e $\boxed{r < n}$ i.e the # of Linearly Indep. Columns in $A = r$, but there are different sets of such r Columns we can combine to reach $\underline{b}$.

i.e we have more Columns than r.

Section 2.2 # 45

(c) exactly one solution for some $\underline{b}$, no solution for other $\underline{b}$.

One $\underline{b}$ can be reached in one way. hence ~~this tells us~~ that # of L.I. Columns in $A = n$.

i.e $\boxed{r=n}$. because if we had more choices of which 'r' columns to select, we would had more solutions than just one.

now, since There are No solutions for other $\underline{b}$, then this means r is Not large enough to span all of $\mathbb{R}^m$. (because $\underline{b}$ lives in $\mathbb{R}^m$).

i.e $\boxed{r<m}$

section 2.2 # 45

(d) exactly one solution for every $\underline{b}$.

since we have a solution for every $\underline{b}$, then A has large enough Column rank (r) to reach all of $\mathbb{R}^m$, which is the space that $\underline{b}$ lives in. hence r must be at least equal to m.

Since there is only <u>one</u> solution, then r must also be the same as number of Columns of A which is n.

hence $r = m = n$ (i.e a square matrix).

Section 2.2 # 50

Complete solution to $Ax = \begin{bmatrix}1\\3\end{bmatrix}$ is $x = \begin{bmatrix}1\\0\end{bmatrix} + c\begin{bmatrix}0\\1\end{bmatrix}$.

Find A.

Solution:

$\underline{b}$ lives in $\mathbb{R}^2$. i.e $m=2$.

$\underline{x}$ lives in $\mathbb{R}^2$. i.e $n=2$

since $\boxed{C \text{ is free variable}}$, and we have 2 equations ($m=2$), then assign 2 different values to C to generate 2 equations in A. Solve the resulting 4 equations for $a_{11}, a_{12}, a_{21}, a_{22}$. as follows:

let $C=1$ $\Rightarrow$ $\begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{bmatrix}\left[\begin{bmatrix}1\\0\end{bmatrix} + c\begin{bmatrix}0\\1\end{bmatrix}\right] = \begin{bmatrix}1\\3\end{bmatrix}$

$\Rightarrow \boxed{\begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}1\\c\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}} \xrightarrow{c=1} \begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}$

now let $C=2$ $\Rightarrow$ $\begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix} \Rightarrow$

$$\boxed{\begin{aligned}a_{11}+a_{12} &= 1\\a_{21}+a_{22} &= 3\\a_{11}+2a_{12} &= 1\\a_{21}+2a_{22} &= 3\end{aligned}}$$

So we have by G.E $\Rightarrow$ $\begin{bmatrix}1&1&0&0\\0&0&1&1\\1&2&0&0\\0&0&1&2\end{bmatrix}\begin{bmatrix}a_{11}\\a_{12}\\a_{21}\\a_{22}\end{bmatrix} = \begin{bmatrix}1\\3\\1\\3\end{bmatrix} \to \begin{bmatrix}1&1&0&0\\0&0&1&1\\0&1&0&0\\0&0&1&2\end{bmatrix}\begin{bmatrix}a_{11}\\a_{12}\\a_{21}\\a_{22}\end{bmatrix} = \begin{bmatrix}1\\3\\0\\3\end{bmatrix}$

exchange R_2, R_3 $\Rightarrow$ $\begin{bmatrix}1&1&0&0\\0&1&0&0\\0&0&1&1\\0&0&1&2\end{bmatrix}\begin{bmatrix}a_{11}\\a_{12}\\a_{21}\\a_{22}\end{bmatrix} = \begin{bmatrix}1\\0\\3\\3\end{bmatrix} \to \begin{bmatrix}1&1&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&1\end{bmatrix}\begin{bmatrix}a_{11}\\a_{12}\\a_{21}\\a_{22}\end{bmatrix} = \begin{bmatrix}1\\0\\3\\0\end{bmatrix} \to$ back sub

$\to a_{22}=0,\ a_{21}=3,\ a_{12}=0,\ a_{11}=1 \Rightarrow \boxed{A = \begin{bmatrix}1 & 0\\3 & 0\end{bmatrix}}$ ✓

Simpler argument in solution key.

verify: $\begin{bmatrix}1&0\\3&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix} \to \begin{bmatrix}\boxed{1}&0\\0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} \Rightarrow \left.\begin{matrix}\text{pivot Var} = x_1\\\text{free Var} = x_2\end{matrix}\right\} \to \begin{matrix}x_1 = 1\\x_2 = \text{any}.\end{matrix} \Rightarrow \underline{x} = \begin{pmatrix}1\\0\end{pmatrix} + c\begin{pmatrix}0\\1\end{pmatrix}$