Solution Key for 6.3

HW # 1, 2, 5, 9 , 18 ← Let's leave #18 out of HW.

#1

$A = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$ $A^TA = \begin{bmatrix} 5 & 20 \\ 20 & 80 \end{bmatrix}$ eigenvalues $(5-\lambda)(80-\lambda) - 400 = 0$

$\lambda^2 - 85\lambda + 400 - 400 = 0$

$\lambda(\lambda - 85) = 0$

hence $\lambda = 0$, $\lambda = 85$.

$\sigma_1^2 = 85$

$A^TA - \lambda I = \begin{bmatrix} -80 & 20 \\ 20 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow -80x_1 + 20x_2 = 0$

$x_2 = 4x_1$ $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

if we normalize, we get $\vec{v}_1 = \boxed{\frac{1}{\sqrt{17}}\begin{bmatrix} 1 \\ 4 \end{bmatrix}}$

$\sigma_2^2 = 0$ $\begin{bmatrix} 5 & 20 \\ 20 & 80 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$5x + 20y = 0$

$x = -4y$ $\vec{v}_2 = \boxed{\frac{1}{\sqrt{17}}\begin{bmatrix} 4 \\ -1 \end{bmatrix}}$

#2

a) $AA^T = \begin{bmatrix} 17 & 34 \\ 34 & 68 \end{bmatrix}$ has $\sigma_1^2 = 85$ and $\sigma_2^2 = 0$

$\vec{u}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\vec{u}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

b) $A\vec{v}_1 = \frac{1}{\sqrt{17}}\begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{17}}\begin{bmatrix} 17 \\ 34 \end{bmatrix} \cdot \frac{\sqrt{17}}{\sqrt{17}} = \sqrt{17}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\sigma_1\vec{u}_1 = \frac{\sqrt{85}}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \sqrt{17}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ✓

$$\underbrace{\begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} \sqrt{85} & 0 \\ 0 & 0 \end{bmatrix}}_{\Sigma}\underbrace{\begin{bmatrix} 1/\sqrt{17} & 4/\sqrt{17} \\ 4/\sqrt{17} & -1/\sqrt{17} \end{bmatrix}}_{V^T} = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}\begin{bmatrix} \sqrt{5} & 4\sqrt{5} \\ 0 & 0 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}}_{A}$$

c) $A = U \Sigma V^T$

Note: this part can be easily answered from Remark 2 in pg 331. I am including lots of detail as explanation.

For the column space of A, let's look at
$$A\vec{x} = U\Sigma V^T\vec{x} = U(\Sigma V^T\vec{x}) = U\vec{c}$$
with $\vec{c} = \Sigma V^T \vec{x} = \begin{bmatrix} \sigma_1 v_1^T \\ 0 \end{bmatrix}$

So the column space of A is spanned by the first column of U.

$C(A)$ is spanned by $\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

For the nullspace of A, we look at the solutions of $A\vec{x} = \vec{0}$.

$$U\Sigma V^T \vec{x} = \vec{0}$$

$U(\Sigma V^T\vec{x}) = \vec{0}$ when $\Sigma V^T \vec{x} = \vec{0}$ since U has orthonormal columns.

$$\Sigma V^T \vec{x} = \vec{0}$$

$\begin{bmatrix} \sigma_1 \vec{v}_1^T \vec{x} \\ 0 \cdot \vec{v}_2^T \vec{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ when $\vec{v}_1^T \vec{x} = 0$, i.e. since the $\vec{v}_i$'s are orthogonal, $\vec{x}$ is in the direction of $\vec{v}_2$.

The nullspace $N(A)$ is spanned by $\frac{1}{\sqrt{17}}\begin{bmatrix} 4 \\ -1 \end{bmatrix}$

The row space is $C(A^T)$ i.e. $C(V\Sigma^T U^T)$ so by the same argument as above it will be spanned by the first column of

$$\frac{1}{\sqrt{17}}\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$N(A^T)$ would be spanned by the second column of U:

$$\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

The columns of U and V^T give us bases for the four fundamental spaces.

5. in selected solutions in text.

9. I want to elaborate on answer in text:

$$A = U\Sigma V^T = U(\Sigma V^T)$$

First we interpret matrix multiplication in the row sense

$$\Sigma V^T = \begin{bmatrix} \sigma_1 \vec{v}_1^{\,T} \\ \sigma_2 \vec{v}_2^{\,T} \\ \vdots \\ \sigma_r \vec{v}_r^{\,T} \\ \vec{0} \end{bmatrix}$$

Note that there are r non-zero singular values, after that the rows are zero.

Then we interpret matrix multiplication in the Column sense

$$U \begin{bmatrix} \sigma_1 \vec{v}_1^{\,T} \\ \sigma_2 \vec{v}_2^{\,T} \\ \vdots \\ \sigma_r \vec{v}_r^{\,T} \\ 0 \end{bmatrix} = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \end{bmatrix} \begin{bmatrix} \sigma_1 \vec{v}_1^{\,T} \\ \sigma_2 \vec{v}_2^{\,T} \\ \vdots \\ \sigma_r \vec{v}_r^{\,T} \\ 0 \end{bmatrix}$$

$$= \sigma_1 \vec{u}_1 \vec{v}_1^{\,T} + \sigma_2 \vec{u}_2 \vec{v}_2^{\,T} + \cdots \sigma_r \vec{u}_r \vec{v}_r^{\,T} \quad \square$$

#18 Explores the conection between SVD and least squares but I don't think we will have time to cover it. It will basically answer the question as to what you do with

$\hat{x} = (A^TA)^{-1}A^T\vec{b}$ when $(A^TA)^{-1}$ does not exist.

The solution will use SVD.