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Not the problems I assigned.

6.32?

HW~~10~~ 11, Math 307. CSUF. Spring 2007.

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Contents

Not the problems I assigned, make sure you look at the solution key.

Hence C is similar to A.

Now for the second part. We write

$$S^{-1}AS = I$$
$$S^{-1}A = S$$

So A must be I, hence only I is similar to I.

2 Section 5.6 problem 2

problem: Describe in words all the matrices that are similar to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and find 2 of them

answer:

Let A be the above matrix. The above matrix represents a reflection across the x-axis. Hence Reflection across the y axis will be similar to it. Any multiple of this reflection matrix will also be similar to A.

Since reflection across the y-axis is $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ then this B matrix is similar to A

Then any multiple of B is also similar to A, such as $\begin{pmatrix} -10 & 0 \\ 0 & 10 \end{pmatrix}$ and $\begin{pmatrix} -20 & 0 \\ 0 & 20 \end{pmatrix}$

3 Section 5.6, problem 5

Problem: show (if B is invertible) then BA is similar to AB

answer: we want to show that $M^{-1}(BA)M = AB$

Let $M^{-1}(BA)M = H$, i.e. let $BA\tilde{}H$, and try to show that $H = AB$

$$\begin{aligned}
M^{-1}(BA)M &= H \\
(BA)M &= MH \\
BA &= MHM^{-1} \\
A &= B^{-1}MHM^{-1} \\
AB &= B^{-1}MHM^{-1}B \\
AB &= \left(B^{-1}M\right)H\left(M^{-1}B\right) \\
AB &= \left(M^{-1}B\right)^{-1}H\left(M^{-1}B\right)
\end{aligned}$$

Let $M^{-1}B = Z$, hence the above becomes

$$AB = Z^{-1}HZ$$

Then $H\tilde{}AB$

But we started by stating that $H\tilde{}BA$, and since if $r_1\tilde{}r_2$ and $r_2\tilde{}r_3$ then $r_1\tilde{}r_3$ then we showed $BA\tilde{}AB$.

4 Section 5.6 problem 18

problem: find normal matrix ($NN^H = N^H N$) that is not Hermitian, skew symmetric, unitary, or diagonal. Show that all permutation matrices are normal

answer:

5 Section 6.1, problem 1

problem: quadratic $f = x^2 + 4xy + 2y^2$ has saddle point at origin, despite that its coefficients are positive. Write f as difference of 2 squares

answer: Let $f = (ax + by)^2 - (cx + dy)^2$, hence

$$\begin{aligned} f &= (ax + by)^2 - (cx + dy)^2 \\ &= a^2x^2 + b^2y^2 + 2abxy - \left(c^2x^2 + d^2y^2 + 2cdxy\right) \\ &= a^2x^2 + b^2y^2 + 2abxy - c^2x^2 - d^2y^2 - 2cdxy \\ &= x^2\left(a^2 - c^2\right) + y^2\left(b^2 - d^2\right) + xy\left(2ab - 2cd\right) \end{aligned}$$

Hence, compare coefficients, we have $a^2 - c^2 = 1, b^2 - d^2 = 2, 2ab - 2cd = 4$

so $ab - cd = 2$.

Let $c = 1$, then we have

$a^2 = 2$,$b^2 - d^2 = 2, 2ab - 2d = 4$

3 equations in 3 unknown. Solve with computer for speed (running out of time!) I get one of the solutions as

$d = 0, a = -\sqrt{2}, b = -\sqrt{2}$

So $f = (ax + by)^2 - (cx + dy)^2 = \boxed{\left(-\sqrt{2}x + -\sqrt{2}y\right)^2 - (x)^2}$

6 Section 6.1, problem 8

problem: decide for or against PD for these matrices, write out corresponding $f = x^T A x$

Answer: I use $a > 0$, and $ac > b^2$ test where $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

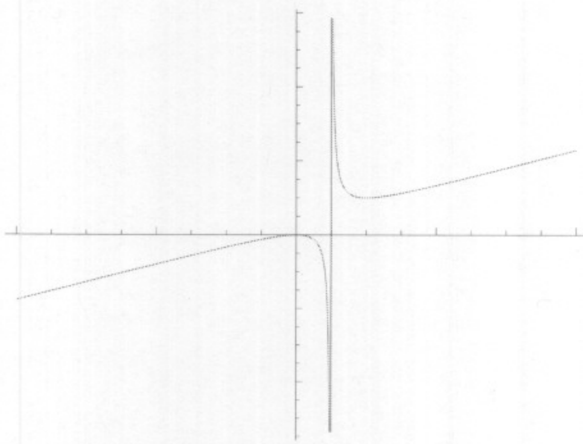
$\begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix} \rightarrow 1 > 0, 5 > 9$ no, $\boxed{\text{Not PD}} \rightarrow f = ax^2 = 2bxy + cy^2 \rightarrow \boxed{f = x^2 + 6xy + 3y}$

$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \rightarrow a > 0, 1 > 1$,no, $\boxed{\text{Not PD}} \rightarrow f = ax^2 = 2bxy + cy^2 \rightarrow \boxed{f = x^2 - 2xy + y}$

$\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \rightarrow a > 0, 10 > 9$,yes, $\boxed{\text{PD}} \rightarrow f = ax^2 = 2bxy + cy^2 \rightarrow \boxed{f = 2x^2 + 6xy + 5y}$

$\begin{pmatrix} -1 & 2 \\ 2 & -8 \end{pmatrix} \rightarrow -1 > 0, no$ $\boxed{\text{Not PD}} \rightarrow f = ax^2 + 2bxy + cy^2 \rightarrow \boxed{f = -x^2 + 4xy - 8y}$

For (b) we have $f = x^2 - 2xy + y$, if $y = \frac{x^2}{2x-1}$ then $f = x^2 - 2x\frac{x^2}{2x-1} + \frac{x^2}{2x-1} = 0$, hence I plot this:



And along the lines shown is $f = 0$

7 Section 6.1, problem 3

problem: if A is 2x2 symmetric matrix, passes test that a>0 , $ac > b^2$ solve equation det $(A - \lambda I) = 0$ and show that eigenvalues are >0

answer:

Matrix is PD, then

$$\begin{aligned} det\left(\begin{pmatrix} a & b \\ b & c \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) &= 0 \\ \left|\begin{pmatrix} a-\lambda & b \\ b & c-\lambda \end{pmatrix}\right| &= 0 \\ (a-\lambda)(c-\lambda) - b^2 &= 0 \\ ac - a\lambda - c\lambda + \lambda^2 &= 0 \\ \lambda^2 + \lambda(-a-c) + ac &= 0 \end{aligned}$$

Hence $\lambda_1 = a, \lambda_2 = c$

But $a > 0$, so $\lambda_1 > 0$, and given $ac >$ positive quantity b^2, then $\lambda_2 = c \rightarrow \lambda_2 > 0$

8 Section 6.1 problem 5

(a) For which numbers b is $\begin{pmatrix} 1 & b \\ b & 9 \end{pmatrix}$ PD?

$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is PD is $a > 0$ and $ac > b^2$

for PD need $ac > b^2$, hence need $9 > b^2$ ie. $b < 3$ and $b > -3$, so $\boxed{-3 < b < 3}$

(b)Factor $A = LDL^T$ when b is in the range above

$\begin{pmatrix} 1 & b \\ b & 9 \end{pmatrix} \to l_{21} = b \to U = \begin{pmatrix} 1 & b \\ 0 & 9 - b^2 \end{pmatrix}$

So $L = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 0 & 9 - b^2 \end{pmatrix}, L^T = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$

(c) What is the minimum of $f(x, y) = \frac{1}{2}(x^2 + 2bxy + 9y^2) - y$ when in this range

when $f(x, y) = \frac{1}{2}(x^2 + 2bxy + 9y^2) - y = \frac{1}{2}x^2 + bxy + \frac{9}{2}y^2 - y$

$\frac{\partial f}{\partial x} = x + by = 0, \frac{\partial f}{\partial y} = bx + 9y - 1 = 0$

Hence $\begin{pmatrix} 1 & b \\ b & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 & b \\ 0 & 9 - b^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Hence $\boxed{y = \frac{1}{9-b^2}}$,and $x + by = 0 \to \boxed{x = -\frac{b}{9-b^2}}$

So $f(x, y) = \frac{1}{2}(x^2 + 2bxy + 9y^2) - y$

Hence $f(x, y) \to \frac{1}{2}\left(\left(-\frac{b}{9-b^2}\right)^2 + 2b\left(-\frac{b}{9-b^2}\right)\left(\frac{1}{9-b^2}\right) + 9\left(\frac{1}{9-b^2}\right)^2\right) - \left(\frac{1}{9-b^2}\right) = \frac{1}{2(b^2-9)}$

So minimum is $\boxed{\frac{1}{2(b^2-9)}}$

(d)When $b = 3$, we see that we get $\frac{1}{0} = \infty$ so no minimum

9 Section 6.1 problem 17

Problem: If A has independent columns then $A^T A$ is square and symmetric and invertible. Rewrite $\vec{x}^T A^T A\vec{x}$ to show why it is positive except when $\vec{x} = 0$, then $A^T A$ is PD

answer: $\vec{x}^T \left(A^T A\right) \vec{x} = (A\vec{x})^T A\vec{x}$ ✓

Let $A\vec{x} = \vec{b}$, then the above is $\vec{b}^T\vec{b} = \left\|\vec{b}\right\|^2$, which is positive quantity except when $\vec{b} = \vec{0}$, which occurs when $A\vec{x} = \vec{b} = \vec{0}$ which happens only when $\vec{x} = \vec{0}$, since A is invertible. ✓

Linearly independent columns, it doesn't need to be invertible.

Hence $A^T A$ is positive definite except when $\vec{x} = 0$

10 Section 6.2, problem 7

problem: If $A = Q\Lambda Q^T$ is P.D. then $R = Q\sqrt{\Lambda}Q^T$ is its S.P.D. square root. Why does R have positive eigenvalues? Compute R and verify $R^2 = A$ for $A = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix}, A = \begin{pmatrix} 10 & -6 \\ -6 & 10 \end{pmatrix}$

answer:

For $A = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix}$

Given R is P.D. (problem said so), Hence $\vec{x}^T R \vec{x} > 0$ for all $\vec{x} \neq 0$

Now (assuming in all that follows that $x \neq 0$)

$$\begin{aligned} R\vec{x} &= \lambda \vec{x} \\ \vec{x}^T R \vec{x} &= x^T \lambda \vec{x} \\ \vec{x}^T R \vec{x} &= \lambda \left\| \vec{x} \right\|^2 \end{aligned}$$

Since $\vec{x}^T R \vec{x} > 0$ then $\lambda \left\| \vec{x} \right\|^2 > 0$, and since $\left\| \vec{x} \right\|^2 > 0$ hence $\boxed{\lambda > 0}$

To compute R we first need to find Q.

$A = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix} \to l_{21} = \frac{6}{10} \to \begin{pmatrix} 10 & 6 \\ 6 - \frac{6}{10} \times 10 & 10 - \frac{6}{10} \times 6 \end{pmatrix} \to \begin{pmatrix} 10 & 6 \\ 0 & \frac{32}{5} \end{pmatrix}$

Hence $L = \begin{pmatrix} 1 & 0 \\ \frac{6}{10} & 1 \end{pmatrix}, U = \begin{pmatrix} 10 & 6 \\ 0 & \frac{32}{5} \end{pmatrix}$

Then

$$\begin{aligned} LDU &= \begin{pmatrix} 1 & 0 \\ \frac{6}{10} & 1 \end{pmatrix} \begin{pmatrix} 10 & 0 \\ 0 & \frac{32}{5} \end{pmatrix} \begin{pmatrix} 1 & \frac{6}{10} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ \frac{6}{10} & 1 \end{pmatrix} \begin{pmatrix} 10 & 0 \\ 0 & \frac{32}{5} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{6}{10} & 1 \end{pmatrix}^T \end{aligned}$$

Hence we see that $Q = L = \begin{pmatrix} 1 & 0 \\ \frac{6}{10} & 1 \end{pmatrix}, \Lambda = D = \begin{pmatrix} 10 & 0 \\ 0 & \frac{32}{5} \end{pmatrix}, Q^T = L^T$

Since A is SPD, then $A = R^T R$ and $A = Q\Lambda Q^T$, hence we can take $R = \sqrt{\Lambda}Q^T$

$$\begin{aligned} R = \sqrt{\Lambda}Q^T &= \sqrt{\begin{pmatrix} 10 & 0 \\ 0 & \frac{32}{5} \end{pmatrix}} \begin{pmatrix} 1 & \frac{6}{10} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{\frac{32}{5}} \end{pmatrix} \begin{pmatrix} 1 & \frac{6}{10} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{10} & \frac{3}{5}\sqrt{2}\sqrt{5} \\ 0 & \frac{4}{5}\sqrt{2}\sqrt{5} \end{pmatrix} \end{aligned}$$

Verify that $R^T R = A$

$$
\begin{aligned}
R^T R &= \begin{pmatrix} \sqrt{10} & 0 \\ \frac{3}{5}\sqrt{2}\sqrt{5} & \frac{4}{5}\sqrt{2}\sqrt{5} \end{pmatrix} \begin{pmatrix} \sqrt{10} & \frac{3}{5}\sqrt{2}\sqrt{5} \\ 0 & \frac{4}{5}\sqrt{2}\sqrt{5} \end{pmatrix} \\
&= \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix}
\end{aligned}
$$

verified oK.

Now do the same for $A = \begin{pmatrix} 10 & -6 \\ -6 & 10 \end{pmatrix}$

$A = \begin{pmatrix} 10 & -6 \\ -6 & 10 \end{pmatrix} \to l_{21} = \frac{-6}{10} \to U = \begin{pmatrix} 10 & -6 \\ -6 - \frac{-6}{10} \times 10 & 10 - \frac{-6}{10} \times -6 \end{pmatrix} \to \begin{pmatrix} 10 & -6 \\ 0 & \frac{32}{5} \end{pmatrix}$:

Hence $L = \begin{pmatrix} 1 & 0 \\ -\frac{6}{10} & 1 \end{pmatrix}, U = \begin{pmatrix} 10 & -6 \\ 0 & \frac{32}{5} \end{pmatrix}$

Then

$$
\begin{aligned}
LDU &= \begin{pmatrix} 1 & 0 \\ -\frac{6}{10} & 1 \end{pmatrix} \begin{pmatrix} 10 & 0 \\ 0 & \frac{32}{5} \end{pmatrix} \begin{pmatrix} 1 & -\frac{6}{10} \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ -\frac{6}{10} & 1 \end{pmatrix} \begin{pmatrix} 10 & 0 \\ 0 & \frac{32}{5} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{6}{10} & 1 \end{pmatrix}^T
\end{aligned}
$$

Hence we see that $Q = L = \begin{pmatrix} 1 & 0 \\ -\frac{6}{10} & 1 \end{pmatrix}, \Lambda = D = \begin{pmatrix} 10 & 0 \\ 0 & \frac{32}{5} \end{pmatrix}, Q^T = L^T$

Then now we find R

Since A is SPD, then $A = R^T R$ and $A = Q\Lambda Q^T$, hence we can take $R = \sqrt{\Lambda} Q^T$

$$
\begin{aligned}
R = \sqrt{\Lambda} Q^T &= \sqrt{\begin{pmatrix} 10 & 0 \\ 0 & \frac{32}{5} \end{pmatrix}} \begin{pmatrix} 1 & -\frac{6}{10} \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{\frac{32}{5}} \end{pmatrix} \begin{pmatrix} 1 & -\frac{6}{10} \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{10} & -\frac{3}{5}\sqrt{2}\sqrt{5} \\ 0 & \frac{4}{5}\sqrt{2}\sqrt{5} \end{pmatrix}
\end{aligned}
$$

Verify that $R^T R = A$

$$
\begin{aligned}
R^T R &= \begin{pmatrix} \sqrt{10} & 0 \\ -\frac{3}{5}\sqrt{2}\sqrt{5} & \frac{4}{5}\sqrt{2}\sqrt{5} \end{pmatrix} \begin{pmatrix} \sqrt{10} & -\frac{3}{5}\sqrt{2}\sqrt{5} \\ 0 & \frac{4}{5}\sqrt{2}\sqrt{5} \end{pmatrix} \\
&= \begin{pmatrix} 10 & -6 \\ -6 & 10 \end{pmatrix}
\end{aligned}
$$

verified oK.

11 Section 6.2, problem 4

Show from the eigenvalues that if A is P.D. so is A^2 and so is A^{-1}

answer:

Given A is PD. Hence Eigenvalues of A are positive.

Let eigenvalue of A be λ_A

Let $B = A^2$

Let eigenvalue of B be λ_B

We need to show that $\lambda_B > 0$

Now

$$\begin{aligned} Bx &= \lambda_B x \\ A^2 x &= \lambda_B x \\ AAx &= \lambda_B x \\ A\lambda_A x &= \lambda_B x \\ \lambda_A A x &= \lambda_B x \\ \lambda_A \lambda_A x &= \lambda_B x \end{aligned}$$

From the last statement above we can now say

$$\lambda_A^2 = \lambda_B \quad \checkmark$$

Hence $\lambda_B > 0$, hence by theorem 6B which says that if all eigenvalues are positive then the matrix is PD, then in this case the matrix B which is A^2 is PD. QED

Now for A^{-1}

$$Ax = \lambda_A x$$

pre multiply both sides by A^{-1}

$$\begin{aligned} \overbrace{A^{-1}A}^{I} x &= A^{-1}\lambda_A x \\ x &= A^{-1}\lambda_A x \\ \frac{1}{\lambda_A} x &= A^{-1} x \end{aligned}$$

i.e.

$$A^{-1} x = \frac{1}{\lambda_A} x \quad \checkmark$$

Hence eigenvalue of A^{-1} is $\frac{1}{\lambda_A}$. And since $\lambda_A > 0$, then so is $\frac{1}{\lambda_A}$, and by theorem 6B again, since all eigenvalues are positive then A^{-1} is P.D.

12 Section 6.2, problem 6

From the pivots, eigenvalues, eigenvectors of $A = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$, write A as $R^T R$ in 3 ways

1. $\left(L\sqrt{D}\right)\left(\sqrt{D}L^T\right)$
2. $\left(Q\sqrt{\Lambda}\right)\left(\sqrt{\Lambda}Q^T\right)$
3. $\left(Q\sqrt{\Lambda}Q^T\right)\left(Q\sqrt{\Lambda}Q^T\right)$

Answer:

First find if A is PD or not. Since this is a 2 by 2 matrix, a simple test is to look at the quantity $a^2 - bc$ and if it is positive, and if a is also positive, then the matrix is PD

$$\begin{aligned} A &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ a &= 5 > 0 \\ a^2 - bc &= 25 - 16 \\ &= 9 > 0 \end{aligned}$$

hence A is P.D.

Then it can be written as $R^T R$ where R is full rank square matrix.

1) Since A is symmetric $P.D.$, then it has choleskly decomposition CC^T where $C = L\sqrt{D}$, and $C^T = \sqrt{D}L^T$ (the pivots are positive in the D matrix diagonal, so we can take their square root)

Then we write $A = R^T R = \left(L\sqrt{D}\right)\left(\sqrt{D}L^T\right)$ where $R = \left(\sqrt{D}L^T\right)$

$\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} \rightarrow l_{21} = \frac{4}{5} \rightarrow U = \begin{pmatrix} 5 & 4 \\ 0 & 5 - \frac{4}{5} \times 4 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 0 & \frac{9}{5} \end{pmatrix}$

Hence $L = \begin{pmatrix} 1 & 0 \\ \frac{4}{5} & 1 \end{pmatrix}, U = \begin{pmatrix} 5 & 4 \\ 0 & \frac{9}{5} \end{pmatrix} \rightarrow LDU = \begin{pmatrix} 1 & 0 \\ \frac{4}{5} & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & \frac{9}{5} \end{pmatrix} \begin{pmatrix} 1 & \frac{4}{5} \\ 0 & 1 \end{pmatrix}$

Hence $R = \sqrt{\begin{pmatrix} 5 & 0 \\ 0 & \frac{9}{5} \end{pmatrix}} \begin{pmatrix} 1 & \frac{4}{5} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{\frac{9}{5}} \end{pmatrix} \begin{pmatrix} 1 & \frac{4}{5} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{5} & \frac{4}{5}\sqrt{5} \\ 0 & \frac{3}{5}\sqrt{5} \end{pmatrix}$

Hence

$$A = \overbrace{\begin{pmatrix} \sqrt{5} & 0 \\ \frac{4}{5}\sqrt{5} & \frac{3}{5}\sqrt{5} \end{pmatrix}}^{L\sqrt{D}} \overbrace{\begin{pmatrix} \sqrt{5} & \frac{4}{5}\sqrt{5} \\ 0 & \frac{3}{5}\sqrt{5} \end{pmatrix}}^{\sqrt{D}L^T}$$

2)From $A = Q\Lambda Q^T$ where Q is the matrix which contains as its columns the normalized eigenvectors of A and Λ contains in its diagonal the eigenvalues of A. First start by finding eigenvalues and eigenvectors of A

13 Section 6.2 problem 8

problem: if A is SPD and C is nonsignular, prove that $B = C^T AC$ is also SPD

solution: Since A is SPD, then it has positive eigenvalues.

Since B is similar to A (given), then B has the same eigenvalues as A, Hence B also has all its eigenvalues positive.

Hence by theorem 6B, B is symmetric positive definite.