Solution Key

5.6 # 1, 2, 5, 11, 18, 29

#1

B is similar to $A \Rightarrow B = M^{-1}AM$

C is similar to $B \Rightarrow C = N^{-1}BN$

Then
$$C = N^{-1}(M^{-1}AM)N$$
$$= N^{-1}M^{-1}AMN$$
$$= (MN)^{-1}A\,MN \quad \text{let } W = MN$$
$$= W^{-1}AW \quad \Rightarrow A \text{ and } C \text{ are similar.}$$

Any matrix that is similar to I

is $B = M^{-1}IM = I$! The only matrix similar to I is I!

#2

$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, all matrices that are similar to A are of

the form $B = M^{-1}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}M$

or equivalently have eigenvalues $1, -1$.

For examples we can either pick an $M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$$M^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$$

Or we could just write any 2x2 matrix with eigenvalues $1, -1$:

$$B_2 = \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix} \quad \text{for example.}$$

#5 Show that if B is invertible, then BA is similar to AB.

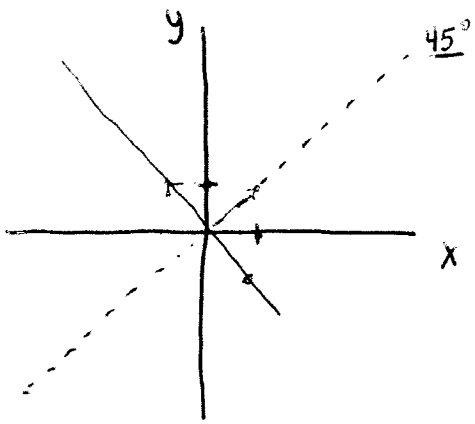
$$B^{-1}B = I$$

$$B^{-1}BA = A$$

$$B^{-1}\underline{BA}B = \underline{AB}$$

So $M^{-1}\,BA\,M = AB$ with $\underline{M=B}$.

11



$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = 0\begin{bmatrix}1\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = 1\begin{bmatrix}1\\0\end{bmatrix} + 0\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$T_v = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$$

The key to getting this matrix correctly is to use the new basis for all components of the computation

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = 1\begin{bmatrix}1\\1\end{bmatrix} + 0\begin{bmatrix}1\\-1\end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = 0\begin{bmatrix}1\\1\end{bmatrix} - 1\begin{bmatrix}1\\-1\end{bmatrix}$$

$$T_V = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$

5Q in text shows how the matrices are similar

$$T_V = M^{-1}\,T_v\,M$$

M takes a vector in V and changes the coordinates to v.

In this case the same vector can be written in two separate bases:

$$a\begin{bmatrix}1\\1\end{bmatrix} + b\begin{bmatrix}1\\-1\end{bmatrix} = x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix}$$

$$\underset{M}{\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}}\underset{\substack{\uparrow\\ \text{vector in}\\ V}}{\begin{bmatrix}a\\b\end{bmatrix}} = \underset{\substack{\uparrow\\ \text{vector}\\ \text{in } v.}}{\begin{bmatrix}x\\y\end{bmatrix}}$$

#18 An example of a matrix that is not Hermitian but is normal is

$$A = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

$$AA^H = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$A^H A = \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

The key point is that you don't have to be Hermitian, skew-Hermitian, unitary or diagonal to be normal (i.e. to be diagonalizable).

I wanted you to show that projection matrices of the form

$p = AA^T$ is normal, since it comes up in SVD. note that this matrix projects onto the column space of A.

$$pp^H = (AA^T)(AA^T)^T = AA^T(A^T)^T A^T$$
$$= AA^T AA^T$$
$$p^H p = (AA^T)^T (AA^T) = (A^T)^T A^T AA^T$$
$$= AA^T AA^T \checkmark$$

So when we do SVD, we will also be able to find enough eigenvectors to diagonalize AA^T.

You could have also shown it was symmetric.

#29 From notes in class we have, if you apply these to the problem, you get the answer in the back of the text.

$$A = M^{-1}\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} M$$

$$A^k = M^{-1}\begin{bmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{bmatrix} M$$

$$e^{At} = M^{-1}\begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} M$$

Section 6.1 #1,2,3,5,17

#1 Solution in text, $f(x,y) = x^2+4xy+2y^2$

$$= (x+2y)^2 - 2y^2,$$

the point is that even if all of the coefficients of a polynomial are positive, the polynomial can still be negative.

#2 In this section, we have matrices of the form $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$

and can use the test for positive definiteness $a>0$ and $ac>b^2$.

a) $A = \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$ $\quad 1>0$, $a>0$ ✓ $\quad ac=5 \not> b^2=9$ $\quad$ not positive definite

$$f(x,y) = [x \ y]\begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x^2+6xy+5y^2$$

b) $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ $\quad 1=a>0$ ✓ $\quad ac = 1 = b^2 \Rightarrow$ not positive definite (actually positive semi-definite)

$$f(x,y) = x^2-2xy+y^2$$

c) $C = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ $\quad a=2>0$ ✓ $\quad ac=10 > b^2=9 \Rightarrow$ <u>positive definite</u>

$$f(x,y) = 2x^2+6xy+5y^2$$

d) $D = \begin{bmatrix} -1 & 2 \\ 2 & -8 \end{bmatrix}$ $\quad a=-1<0$ X $\quad$ Not positive definite.

$$f(x,y) = -x^2+4xy-8y^2$$

For part b) if we write $f(x,y)$ as a sum of squares

$$x^2-2xy+y^2 = (x-y)^2$$

so this function will be zero along $x-y=0 \Rightarrow y=x$. you see it it is not posite definite, but positive semi-definite as it always increases or stays zero (along $x=y$).

#3. Solution in text. The point is to make a connection with the next section where we show that having all positive eigenvalues is equivalent to being positive definite.

#5. The solution is in the text.

A couple of comments that might be useful:

c) to find the minimum value of $f(x,y)$ first you find the critical point (x_c, y_c) so thad $\frac{\partial}{\partial x} f(x_c, y_c) = \frac{\partial f}{\partial y}(x_c, y_c) = 0$ and then you plug those values (x_c, y_c) to find the minimum of $f = f(x_c, y_c)$.

d) If $b=3$, $f(x,y) = \frac{1}{2}(x^2 + 6xy + 9y^2) - y$

$$= \frac{1}{2}(x+3y)^2 - y$$

if we look at the line $x+3y=0 \Rightarrow x = -3y$

then as $y \to \infty$, $f(x,y) \to 0 - \infty$ and hence $f(x,y)$ has no minimum.

The key is that when you face a non-invertible matrix in the calculation (like when $b=3$), you go back to basics.

#17 A^TA is important for SVD. You see how I pick problems :).

$\vec{x}^T A^T A \vec{x} = \underset{R}{(A\vec{x})}^T \underset{R}{(A\vec{x})} = \|A\vec{x}\|^2 = 0$ only if $A\vec{x} = \vec{0}$

Since A has linearly independent columns that only happens when $\vec{x} = 0$.

6.2 # 4, 6, 7, 8

4. If A is positive definite then it has positive eigenvalues.

$A\vec{x} = \lambda\vec{x}$ with $\lambda > 0$

to look at the eigenvalues of A^2 we multiply the eigenvalue equation by A

$$A^2\vec{x} = \lambda A\vec{x}$$
$$= \lambda^2\vec{x} \Rightarrow$$ A^2 has eigenvalues λ^2 hence is also positive & A^2 is positive definite.

$$A^{-1}A\vec{x} = \lambda A^{-1}\vec{x}$$
$$\vec{x} = \lambda A^{-1}\vec{x}$$
$$\frac{1}{\lambda}\vec{x} = A^{-1}\vec{x} \Rightarrow$$ the eigenvalues of A^{-1} are $\frac{1}{\lambda}$ which are positive if λ is positive.

#6

$A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$ for $(L\sqrt{D})(\sqrt{D}L^T) \Rightarrow R = \sqrt{D}L^T = \frac{1}{\sqrt{5}}\begin{bmatrix} 5 & 4 \\ 0 & 3 \end{bmatrix}$

$A = R^TR$ $\quad (Q\sqrt{\Lambda})(\sqrt{\Lambda}Q^T) \Rightarrow R = \sqrt{\Lambda}Q^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 3 & 3 \end{bmatrix}$

$(Q\sqrt{\Lambda}Q^T)(Q\sqrt{\Lambda}Q^T) \Rightarrow R = Q\sqrt{\Lambda}Q^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

The point is that R is not unique but as long as you can write $A = R^TR$ with R having linearly independent columns, A is positive definite.

#7 Solution in text, but I elaborate.

$$A = R^2 = (Q\sqrt{\Lambda}Q^T)(Q\sqrt{\Lambda}Q^T)$$

the eigenvalues of R are $\sqrt{\lambda_i}$ since $R = Q\sqrt{\Lambda}Q^T$

and since R^2 has eigenvalues λ_i (by Problem 4) we see that $A\vec{x} = \lambda_i\vec{x}$ is P.D. then $\lambda_i > 0$ and $\sqrt{\lambda_i} > 0$.

#8 A is symmetric positive definite, C is invertible, we need to prove $B = C^TAC$ is symmetric positive definite.

Symmetric $(C^TAC)^T = C^TA^TC$
$= C^TAC$ ✓

P.D. let $A = R^TR$, then $B = C^TR^TRC$,
let $W = RC$
$B = W^TW$

$RC\vec{x} = \vec{0}$ only if $C\vec{x} = \vec{0}$ since R has linearly independent columns & $C\vec{x} = \vec{0}$ when $\vec{x} = \vec{0}$ since C is invertible.

Since $RC\vec{x} = \vec{0}$ only has $\vec{x} = \vec{0}$ as a solution,