Solution Key HW #10 Math 307

Section 5.4 # 1,2,3,9

#1 Solution in text.

• #2.

$\frac{d\vec{u}}{dt} = A\vec{u}$, has a general solution $\vec{u}(t) = c_1 e^{-2t}\vec{x}_1 + c_2\vec{x}_2$ ($\vec{x}_1, \vec{x}_2$ from #1)

$\vec{u}(t) = \begin{bmatrix} e^{-2t}+2 \\ -e^{-2t}+2 \end{bmatrix}$ setting $\vec{u}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

As $t \to \infty$, $\vec{u}(t) \to \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. This is the steady state, i.e. the solution as $t \to \infty$.

#3. $u(t) = \begin{bmatrix} e^{2t}+2 \\ -e^{2t}+2 \end{bmatrix}$; Even though the computations are the same as in #2, the result blows up as $t \to \infty$. This differential equation is unstable.

• #9 Solution in text. Note that to be unstable all we need is for the real part of <u>one</u> eigenvalue to be positive.

Section 5.5 #1, 11, 12, 27, 48

#1 Solution in text

• #11 Solution in text. cool eh? any symmetric matrix can be written as a sum of projection matrices.

#12

a) The key for these problems is to realize that if a matrix has no zero eigenvalues, then it is invertible. The simplest way to see this is using the fact that the determinant is the product of the eigenvalues or diagonalizing the matrix if it is diagonalizable.

If A is Hermitian, then it's eigenvalues are real. In a similar way to Problem # 24 (Section 5.1), the eigenvalues of $(A+iI)\vec{x} = (\lambda+i)\vec{x}$ are $\lambda+i$ where λ is an eigenvalue of A. Since λ is real, then $\lambda+i$ cannot be zero and $A+iI$ is invertible $\Rightarrow$ True.

b) The eigenvalues of Q have $|\lambda| = 1$, so $\lambda + \frac{1}{2}$ cannot be zero $\Rightarrow$ True

c) A can have $-i$ as an eigenvalue even if A is real, so $A+iI$ is not necessarily invertible:

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $A+iI = \begin{bmatrix} i & -1 \\ 1 & -i \end{bmatrix}$ $\det(A+iI) = 1-1=0$. Not invertible.

#27 Solution in text.

• #48 We know that A is Hermitian, i.e. $A^H = A$.

We want to prove that A^{-1} is Hermitian, i.e. $(A^{-1})^H = A^{-1}$.

We start with the definition of an inverse

$$A^{-1}A = I$$

Take the conjugate transpose of both sides

$$(A^{-1}A)^H = I^H$$

$$A^H (A^{-1})^H = I$$

$$A (A^{-1})^H = I \quad \text{since } A^H = A.$$

since $AB = I$, $B = A^{-1}$. [For square matrices a right inverse is also the left inverse, and it must be square to be Hermitian.]

so $(A^{-1})^H = A^{-1}$ ∎

equivalently

$$A^{-1} A (A^{-1})^H = A^{-1}$$

$$(A^{-1})^H = A^{-1}.$$