University Course

EGEE 518
Digital Signal Processing I

California State University, Fullerton

Fall 2008

My Class Notes

Nasser M. Abbasi

Fall 2008

Contents

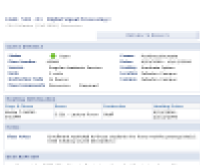
Chapter 1

introduction

I took this course in Fall 2008 at CSUF to learn more about DSP.

This course was hard. The textbook was not too easy, The instructor Dr Shiva has tremendous experience in this subject, and he would explain some difficult things with examples on the board which helped quite a bit. The final exam was hard, it was 7 questions and I had no time to finish them all. It is a very useful course to take to learn about signal processing.

Instructor is professor Shiva, Mostaf, Dept Chair, EE, CSUF.



Chapter 2

Final project

final project

Chapter 3

Study notes

3.1 DSP notes

For fourier transform in mathematica, use these options

In[8]:= **FourierTransform[1, t, s, FourierParameters → {-1, 1}]**

Out[8]= DiracDelta[s]

From Wikipedia. Discrete convolution

Discrete convolution [edit]

For complex-valued functions f, g defined on the set of integers, the **discrete convolution** of f and g is given by:

$$(f*g)[n] \stackrel{\text{def}}{=} \sum_{m=-\infty}^{\infty} f[m]\cdot g[n-m]$$

$$= \sum_{m=-\infty}^{\infty} f[n-m]\cdot g[m]. \quad \text{(commutativity)}$$

Autocorrelaton

energy. Signals that "last forever" are treated instead as random processes, in which case different definitions are needed, based on expected values. For wide-sense-stationary random processes, the autocorrelations are defined as

$$R_{ff}(\tau) = \mathrm{E}\left[f(t)\overline{f}(t-\tau)\right]$$

$$R_{xx}(j) = \mathrm{E}\left[x_n \overline{x}_{n-j}\right].$$

For processes that are not stationary, these will also be functions of t, or n.

For processes that are also ergodic, the expectation can be replaced by the limit of a time average. The autocorrelation of an ergodic process is sometimes defined as or equated to[3]

$$R_{ff}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t+\tau)\overline{f}(t)\,dt$$

$$R_{xx}(j) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} x_n \overline{x}_{n-j}.$$

These definitions have the advantage that they give sensible well-defined single-parameter results for periodic functions, even when those functions are not the output of stationary ergodic processes.

```
function nma_show_fourier

t=-4:.1:4;
N=4;
T=2;
```

```
plot(t,y(t,-N,N,T));


end

%------------------
function v=c(k,T)
term=pi*k/2;
v=(1/T)*sin(term)/term;
end

%---------------------
function v=y(t,from,to,T)

coeff=zeros(to-from+1,1);
k=0;
for i=from:to
    k=k+1;
    coeff(k)=c(i,T);
end

v=zeros(length(t),1);
for i=1:length(t)
    v(i)=0;
    for k=from:to
        v(i)=v(i)+coeff(k)*exp(sqrt(-1)*2*pi/T*k*t(i));
    end
end
end
```

Chapter 4

HWs

4.1 HW2

Local contents

4.1.1 Problem 1

Compute an appropriate sampling rate and DFT size $N = 2^v$ to analyze a single with no significant frequency content above $10khz$ and with a minimum resolution of 100 hz

4.1.1.1 Solution

From Nyquist sampling theory we obtain that sampling frequency is

$$f_s = 20000\ hz$$

Now, the frequency resolution is given by

$$\Delta f = \frac{f_s}{N}$$

where N is the number of FFT samples. Now since the minimum Δf is 100 hz then we write

$$\frac{f_s}{N} = \Delta f \geq 100$$

or

$$\frac{f_s}{N} \geq 100$$

Hence

$$\begin{aligned} N &\leq \frac{20,000}{100} \\ &\leq 200 \text{ samples} \end{aligned}$$

Therefore, we need the closest N below 200 which is power of 2, and hence

$$\boxed{N = 128}$$

4.1.2 Problem 2

sketch the locus of points obtained using Chirp Z Transform in the Z plane for $M = 8, W_0 = 2, \phi_0 = \frac{\pi}{16}, A_0 = 2, \theta_0 = \frac{\pi}{4}$

Answer:

Chirp Z transform is defined as

$$X(z_k) = \sum_{n=0}^{N-1} x[n] z_k^{-n} \qquad k = 0, 1, \cdots, M-1 \tag{1}$$

Where

$$z_k = AW^{-k}$$

and $A = A_0 e^{j\theta_0}$ and $W = W_0 e^{-j\phi_0}$

Hence

$$\begin{aligned} z_k &= \left(A_0 e^{j\theta_0}\right)\left(W_0 e^{-j\phi_0}\right)^{-k} \\ &= \frac{A_0}{W_0^k} e^{j(\theta_0 + k\phi_0)} \end{aligned}$$

Hence

$$\begin{aligned} |z_k| &= \frac{A_0}{W_0^k} \\ &= \frac{2}{2^k} \end{aligned}$$

and

$$\begin{aligned} \textit{phase of } z_k &= \theta_0 + k\phi_0 \\ &= \frac{\pi}{4} + k\frac{\pi}{16} \end{aligned}$$

Hence

k	$\lvert z_k\rvert = \frac{2}{2^k}$	$\textit{phase of } z_k = \frac{\pi}{4} + k\frac{\pi}{16}$	$\textit{phase of } z_k \textit{ in}$ degrees
0	$\frac{2}{1} = 2$	$\frac{\pi}{4} + 0 \times \frac{\pi}{16} = \frac{\pi}{4}$	45
1	$\frac{2}{2} = 1$	$\frac{\pi}{4} + 1 \times \frac{\pi}{16} = \frac{5}{16}\pi$	56.25
2	$\frac{2}{4} = \frac{1}{2}$	$\frac{\pi}{4} + 2 \times \frac{\pi}{16} = \frac{3}{8}\pi$	67.5
3	$\frac{2}{8} = \frac{1}{4}$	$\frac{\pi}{4} + 3 \times \frac{\pi}{16} = \frac{7}{16}\pi$	78.75
4	$\frac{2}{16} = \frac{1}{8}$	$\frac{\pi}{4} + 4 \times \frac{\pi}{16} = \frac{1}{2}\pi$	90
5	$\frac{2}{32} = \frac{1}{16}$	$\frac{\pi}{4} + 5 \times \frac{\pi}{16} = \frac{9}{16}\pi$	101.25
6	$\frac{2}{64} = \frac{1}{32}$	$\frac{\pi}{4} + 6 \times \frac{\pi}{16} = \frac{5}{8}\pi$	112.5
7	$\frac{2}{128} = \frac{1}{64}$	$\frac{\pi}{4} + 7 \times \frac{\pi}{16} = \frac{11}{16}\pi$	123.75

```
In[]:= z[k_, W0_, A0_, θ0_, ϕ0_] := A0 Exp[I θ0] (W0 Exp[-I ϕ0])^-k
W0 = 2;
A0 = 2;
θ0 = Pi / 4;
ϕ0 = Pi / 16;
m = 8;
zValues = Table[z[k, W0, A0, θ0, ϕ0], {k, 0, m - 1}];
arg = Arg[zValues]
abs = Abs[zValues]
data = Transpose[{arg, abs}];
p1 = ListPolarPlot[data, AxesOrigin → {0, 0}, PlotRange → All, Joined → False, PlotMarkers → Automatic,
      PlotStyle → Red];
p2 = ListPolarPlot[data, AxesOrigin → {0, 0}, PlotRange → All, Joined → True];
p3 = PolarPlot[1, {t, 0, 2 Pi}];
Show[p1, p2, p3]
```

Out[]= $\left\{\frac{\pi}{4}, \frac{5\pi}{16}, \frac{3\pi}{8}, \frac{7\pi}{16}, \frac{\pi}{2}, \frac{9\pi}{16}, \frac{5\pi}{8}, \frac{11\pi}{16}\right\}$

Out[]= $\left\{2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}\right\}$

Out[]=

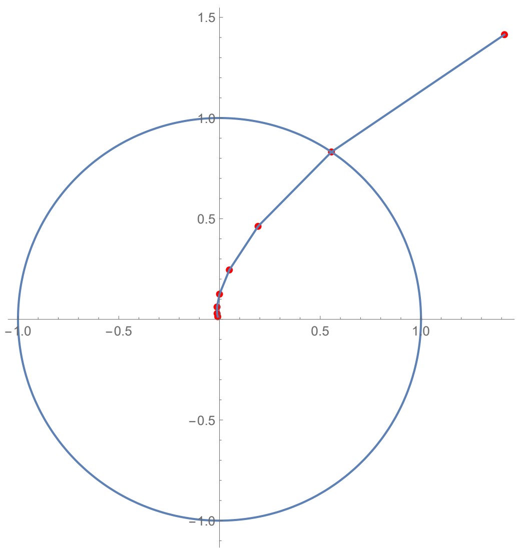


Figure 4.1: plot of the above contour

This is Mathematica notebook used to make plot of the Chirp Z transform contour. This is my graded HW2

4.1.3 graded HW2

HW2, EGEE 518. CSUF, Fall 2008

Nasser Abbasi

October 11, 2008

1 Problem 1

Compute an appropriate sampling rate and DFT size $N = 2^v$ to analyze a single with no significant frequency content above $10khz$ and with a minimum resolution of $100hz$

Solution

From Nyquist sampling theory we obtain that sampling frequency is

$$f_s = 20000\ hz$$

Now, the frequency resolution is given by

$$\Delta f = \frac{f_s}{N}$$

where N is the number of FFT samples. Now since the minimum Δf is 100 hz then we write

$$\frac{f_s}{N} = \Delta f \geq 100$$

or

$$\frac{f_s}{N} \geq 100$$

Hence

$$\begin{aligned} N &\leq \frac{20,000}{100} \\ &\leq 200 \text{ samples} \end{aligned}$$

Therefore, we need the closest N below 200 which is power of 2, and hence

$$\boxed{N = 128}$$ 256

2 Problem 2

sketch the locus of points obtained using Chirp Z Transform in the Z plane for $M = 8, W_0 = 2, \phi_0 = \frac{\pi}{16}, A_0 = 2, \theta_0 = \frac{\pi}{4}$

Answer:

Chirp Z transform is defined as

$$X(z_k) = \sum_{n=0}^{N-1} x[n] z_k^{-n} \qquad k = 0, 1, \cdots, M-1 \tag{1}$$

Where

$$z_k = AW^{-k}$$

and $A = A_0 e^{j\theta_0}$ and $W = W_0 e^{-j\phi_0}$

Hence

$$\begin{aligned} z_k &= \left(A_0 e^{j\theta_0}\right)\left(W_0 e^{-j\phi_0}\right)^{-k} \\ &= \frac{A_0}{W_0^k} e^{j(\theta_0 + k\phi_0)} \end{aligned}$$

Hence

$$\begin{aligned} |z_k| &= \frac{A_0}{W_0^k} \\ &= \frac{2}{2^k} \end{aligned}$$

and

$$\begin{aligned} phase\ of\ z_k &= \theta_0 + k\phi_0 \\ &= \frac{\pi}{4} + k\frac{\pi}{16} \end{aligned}$$

Hence

k	$\lvert z_k \rvert = \frac{2}{2^k}$	$phase\ of\ z_k = \frac{\pi}{4} + k\frac{\pi}{16}$	$phase\ of\ z_k\ in$ degrees
0	$\frac{2}{1} = 2$	$\frac{\pi}{4} + 0 \times \frac{\pi}{16} = \frac{\pi}{4}$	45
1	$\frac{2}{2} = 1$	$\frac{\pi}{4} + 1 \times \frac{\pi}{16} = \frac{5}{16}\pi$	56.25
2	$\frac{2}{4} = \frac{1}{2}$	$\frac{\pi}{4} + 2 \times \frac{\pi}{16} = \frac{3}{8}\pi$	67.5
3	$\frac{2}{8} = \frac{1}{4}$	$\frac{\pi}{4} + 3 \times \frac{\pi}{16} = \frac{7}{16}\pi$	78.75
4	$\frac{2}{16} = \frac{1}{8}$	$\frac{\pi}{4} + 4 \times \frac{\pi}{16} = \frac{1}{2}\pi$	90
5	$\frac{2}{32} = \frac{1}{16}$	$\frac{\pi}{4} + 5 \times \frac{\pi}{16} = \frac{9}{16}\pi$	101.25
6	$\frac{2}{64} = \frac{1}{32}$	$\frac{\pi}{4} + 6 \times \frac{\pi}{16} = \frac{5}{8}\pi$	112.5
7	$\frac{2}{128} = \frac{1}{64}$	$\frac{\pi}{4} + 7 \times \frac{\pi}{16} = \frac{11}{16}\pi$	123.75

2

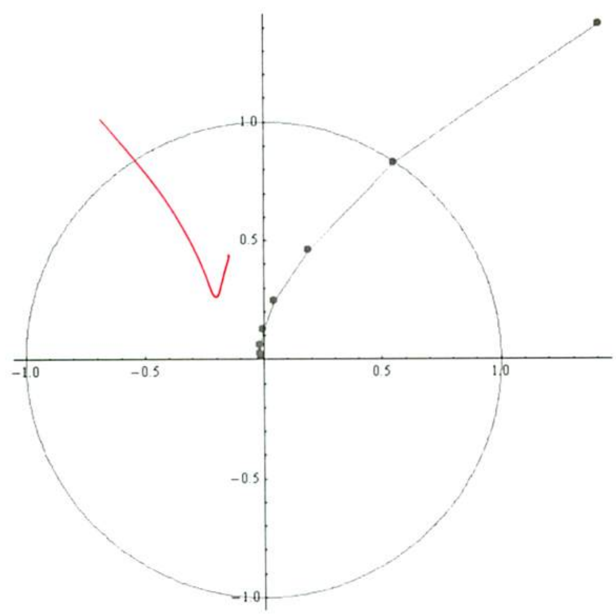
Below is plot of the above contour

```
In[579]:= z[k_, W0_, A0_, θ0_, ϕ0_] := A0 Exp[I θ0] (W0 Exp[-I ϕ0])^-k
W0 = 2;
A0 = 2;
θ0 = Pi / 4;
ϕ0 = Pi / 16;
m = 8;
zValues = Table[z[k, W0, A0, θ0, ϕ0], {k, 0, m - 1}];
arg = Arg[zValues]
abs = Abs[zValues]
data = Transpose[{arg, abs}];
p1 = ListPolarPlot[data, AxesOrigin → {0, 0},
   PlotRange → All, Joined → False,
   PlotMarkers → {Automatic, Automatic}];
p2 = ListPolarPlot[data, AxesOrigin → {0, 0},
   PlotRange → All, Joined → True];
p3 = PolarPlot[1, {t, 0, 2 Pi}];
Show[p1, p2, p3]
```

Out[586]= $\left\{\frac{\pi}{4}, \frac{5\pi}{16}, \frac{3\pi}{8}, \frac{7\pi}{16}, \frac{\pi}{2}, \frac{9\pi}{16}, \frac{5\pi}{8}, \frac{11\pi}{16}\right\}$

Out[587]= $\left\{2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}\right\}$

Out[592]=



3

4.2 HW3

Local contents

4.2.1 my solution

(1)

Definitions

(1) auto correlation $R_{xx}(n,n+m)$: Measures the similarity of R.P. $X(t)$ at time n and $X(t)$ at later time $n+m$.

$$R_{xx}(n,n+m) = E\{X(n)\ X^*(n+m)\}$$

(2) stationary process.

This is a random process whose statistics do not change with shift in time origin.

(3) Wide Sense Stationary process:

This is a random process $X(t)$ which statisfies the following conditions:

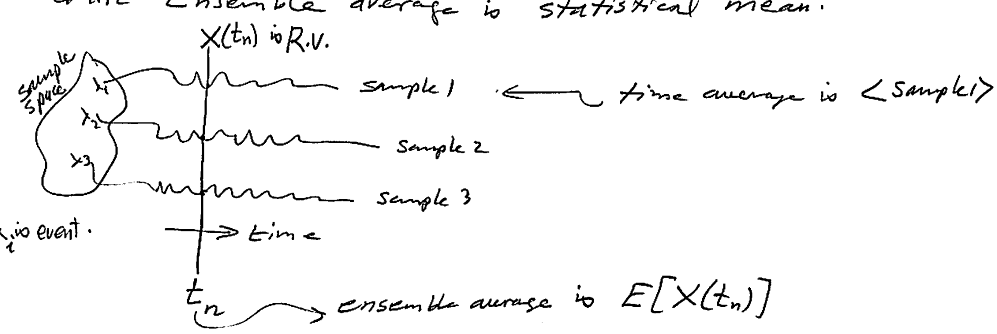
1. its mean is constant. i.e $E[X] =$ constant.
2. autocorrelation depends only on time interval m.

i.e $R_{xx}(n,n+m) = R_{xx}(m)$.

Notice that stationary process is WSS, but WSS is not neccessarly stationary i.e

(4) Time averages, Ensemble averages

Time averages is the average of the sample sequence. while Ensemble average is statistical mean.

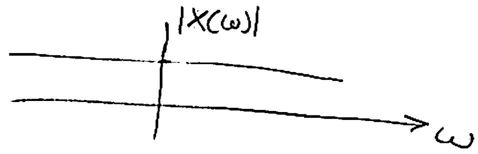


(2)

(5) White Noise;

this is a R.P. whose power spectral density is constant. i.e. power contained in a frequency bandwidth B is the same regardless of where this bandwidth is centered.

"flat" spectrum implies $X(t)$ is white Noise process.



the above is a description in the frequency domain. in the time domain, $\Phi_{XX}(m) = \delta(m)$ i.e. the autocorrelation is nonzero only if time interval is zero. i.e $X(t)$ only correlates with itself at zero time delay. so all R.V. that belong to a white noise process are uncorrelated with each other if time interval is nonzero.

(6) Erogdic Process;

this is a R.P. whose statistics taken from the time samples are the same as statistics taken from Ensembles.

for example. we say a process is Ergodic in the mean, then

$$E\{X(t)\} = \langle X(t) \rangle$$

statisical sample. expected value of R.V.

time average.. mean of a sample (or time series)

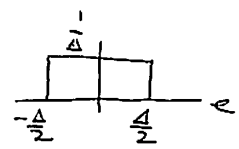
the above equality is in the limit, i.e as the time series length increases. and the statiscal mean is when the Number of time serieses increases as well.

#3 (3)

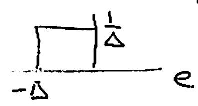
$y(n) = Q[x(n)] = x(n) + e(n)$ → quantization Error

$e(n)$ is white Noise.

Pdf for rounding is uniform



Pdf for truncation is



a) Find mean and Variance due to rounding

b) " " " " " " truncation.

Answer

a) $$m_e = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e \cdot f(e)\, de. = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \frac{e}{\Delta} de = \frac{1}{\Delta}\left(\frac{e^2}{2}\right)_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}}$$

$$= \frac{1}{2\Delta}\left[\left(\frac{\Delta}{2}\right)^2 - \left(-\frac{\Delta}{2}\right)^2\right] = \frac{1}{2\Delta}(0) = \boxed{0}$$

$$E[e^2] = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e^2 f(e)\, de = \frac{1}{\Delta}\int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e^2 de = \frac{1}{\Delta}\left(\frac{e^3}{3}\right)_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}}$$

$$= \frac{1}{3\Delta}\left[\left(\frac{\Delta}{2}\right)^3 - \left(-\frac{\Delta}{2}\right)^3\right] = \frac{1}{3\Delta}\left[\frac{\Delta^3}{8} + \frac{\Delta^3}{8}\right] = \frac{1}{3\Delta}\left[\frac{\Delta^3}{4}\right]$$

$$= \boxed{\frac{\Delta^2}{12}}$$

so $\sigma^2 = E[e^2] - (E[e])^2 = \frac{\Delta^2}{12} - 0^2 = \boxed{\frac{\Delta^2}{12}}$

b) $$m_e = \int_{-\Delta}^{0} e f(e)\, de = \int_{-\Delta}^{0} e \frac{1}{\Delta} de = \frac{1}{\Delta}\left(\frac{e^2}{2}\right)_{-\Delta}^{0} = \frac{1}{2\Delta}\left(0^2 - (-\Delta)^2\right)$$

$$= \frac{1}{2\Delta}\left(0 - \Delta^2\right) = \boxed{-\frac{\Delta}{2}}$$

$$E[e^2] = \int_{-\Delta}^{0} e^2 f(e) de = \int_{-\Delta}^{0} e^2 \frac{1}{\Delta} de = \frac{1}{\Delta}\left[\frac{e^3}{3}\right]_{-\Delta}^{0}$$

$$= \frac{1}{3\Delta}\left[0^3 - (-\Delta)^3\right] = \boxed{\frac{\Delta^2}{3}}$$

so $\sigma^2 = E[e^2] - (E[e])^2 = \frac{\Delta^2}{3} - \left(-\frac{\Delta}{2}\right)^2 = \frac{\Delta^2}{3} - \frac{\Delta^2}{4} = \frac{4\Delta^2 - 3\Delta^2}{12} = \boxed{\frac{\Delta^2}{12}}$

\# 4 let $e(n)$ white Noise sequence. Let $s(n)$ uncorrelated sequence to $e(n)$. Show that $y(n) = s(n)\,e(n)$ is white. i.e $E[\,y(n)\;y(n+m)\,] = A\,\delta(m)$. (4)

Answer

$$E[\,y(n)\;y(n+m)\,] = E\left[\,s(n)e(n)\quad s(n+m)e(n+m)\,\right]$$

$$= E\left[\,s(n)s(n+m)\quad e(n)e(n+m)\,\right]$$

since $e(n)$ and $s(n)$ are uncorrelated, hence independent, then we can write the above as

$$= E\left[\,s(n)\,s(n+m)\,\right]\;E\left[\,e(n)\,e(n+m)\,\right]$$

but $e(n)$ is white. hence $\Phi_{ee}(n,m) = E[\,e(n)\;e(n+m)\,] = \boxed{\delta(m)}$ by definition of white signal.

hence $\Phi_{yy}(n,m) = E\left[\,S(n)\;S(n+m)\,\right]\;\delta(m)$.

Now, when $m=0$, $\Phi_{yy}(n,m) = E[\,S(n)\;S(n)\,]\cdot 1$.

Since $S(n)$ is uncorrelated with white Noise, then $m_s = 0$
Since $S(n)$ is also white.
hence $E[\,s^2(n)\,] =$ Total average power in $S(n)$
$= A$ some Constant.

hence when $m=0$, $\Phi_{yy}(n,m) = A$
when $m \neq 0$ $\Phi_{yy}(n,m) = E[\,S(n)\;S(n+m)\,]\cdot 0$
$= 0$

Therefor $\boxed{\Phi_{yy}(n,m) = A\,\delta(m)}$

Since $\Phi_{yy}(n,m)$ is function of only m, it is white signal.

#6 Consider 2 real stationary random processes $\{x_n\}$ and $\{y_n\}$, with mean m_x, m_y, and variance σ_x^2, σ_y^2. (5)

(a) $\gamma_{xx}(m)$ this is autocovariance.

$$\gamma_{xx}(m) = E\{(x(n)-m_x)(x^*(n+m)-m_x^*)\}$$
$$= E\{x(n)x^*(n+m) - m_x x(n) - m_x x^*(n+m) + m_x^2\}$$
$$= E\{x(n)x^*(n+m)\} - m_x E\{x(n)\} - m_x E\{x^*(n+m)\} + m_x^2.$$
$$= \phi_{xx}(n,n+m) - m_x^2 - m_x E\{x^*(n+m)\} + \cancel{m_x^2}$$
$$= \phi_{xx}(n,n+m) - m_x E\{x^*(n+m)\}.$$

but $\{x_n\}$ is stationary, so its statistics do not change with shift of time origin. hence $E\{x^*(n+m)\} = E\{x^*(n)\} = m_x$.
so above becomes

$$\gamma_{xx}(m) = \phi_{xx}(n,n+m) - m_x^2.$$

but $\phi_{xx}(n,n+m) = \phi_{xx}(m)$ since stationary hence

$$\boxed{\gamma_{xx}(m) = \phi_{xx}(m) - m_x^2}$$

$$\gamma_{xy}(m) = E[(x(n)-m_x)(y^*(n+m)-m_y^*)]$$
$$= E[x(n)y^*(n+m) - m_y^* x(n) - m_x y^*(n+m) + m_y^* m_x]$$
$$= E\{x(n)y^*(n+m)\} - m_y^* E\{x(n)\} - m_x E\{y^*(n+m)\} + m_y^* m_x$$

but due to stationarity, $E\{y^*(n+m)\} = m_y$. so above becomes

$$= E\{x(n)y^*(n+m)\} - m_y m_x - m_x m_y + m_y m_x$$
$$= E\{x(n)y^*(n+m)\} - m_y m_x.$$

but $E\{x(n)y^*(n+m)\} = \phi_{xy}(m)$ since stationary

so $\boxed{\gamma_{xy}(m) = \phi_{xy}(m) - m_y m_x}$ $\rightarrow$

6

(b) $\phi_{xx}(0) = E\{x(n)\,x^*(n+m)\}$

but $m=0$. hence

$\phi_{xx}(0) = E\{x(n)\,x^*(n)\} = E\{x^2(n)\}$.

= mean square.

$\gamma_{xx}(0) = E\{(x(n)-m_x)(x^*(n+m)-m_x^*)\}$

but $m=0$ so

$$
\begin{aligned}
\gamma_{xx}(0) &= E\{(x(n)-m_x)(x^*(n)-m_x^*)\}\\
&= E\{x^2(n) - x(n)m_x - m_x x^*(n) + m_x^2\}\\
&= E\{x^2(n)\} - m_x E\{x(n)\} - m_x E\{x^*(n)\} + m_x^2\\
&= E\{x^2(n)\} - m_x^2 - m_x^2 + \cancel{m_x^2}\\
&= E\{x^2(n)\} - m_x^2
\end{aligned}
$$

but this is the definition of σ_x^2. hence

$$\boxed{\gamma_{xx}(0) = \sigma_x^2}$$

$\longrightarrow$

(c) $\phi_{xx}(m) = E\{x(n)\,x^*(n+m)\} = E\{x^*_{n+m}\,x_n\} = \left(E\{x_{n+m}x^*_n\}\right)^*$ 7

$= \Phi^*_{xx}(-m)$

if process is real, then $\Phi^*_{xx}(-m) = \Phi_{xx}(-m)$.

i.e. $\Phi_{xx}(m) = \Phi_{xx}(-m)$

$\gamma_{xx}(m) = E\{(x(n)-m_x)\,(x^*(n+m)-m^*_x)\}$

$= \Phi_{xx}(m) - m_x m^*_x$ (from part (a)). 1

$= \Phi^*_{xx}(-m) - m_x m^*_x.$ (using result above).

$= \left(E\{x_{n+m}X^*_n\}\right)^* - m_x m^*_x$

$= E\{X^*_{n+m}X_n\} - m_x m^*_x$

$= \left(E\{x_{n+m}\,x^*_n\} - m^*_x\,m_x\right)^*$

$= \gamma^*_{xx}(-m)$

if Real process, then $\gamma^*_{xx}(-m) = \gamma_{xx}(-m)$. $\Longrightarrow$ $\boxed{\gamma_{xx}(m)=\gamma_{xx}(-m)}$

$\Phi_{xy}(m) = E\{(x(n)-m_x)\,(y(n+m)-m_y)\}$

$= E\{x(n)\,y(n+m) - m_y x(n) - m_x y(n+m) + m_x m_y\}$

$= E\{x(n)\,y(n+m)\} - m_y m_x - m_x m_y + m_x m_y$

$= E\{x(n)\,y(n+m)\} - m_y m_x$

But $\Phi_{yx}(-m) = E\{(y(n)-m_y)\,(x(n-m)-m_x)\}$

$= E\{y(n)\,x(n-m)\} - m_x m_y - m_y m_x + m_y m_x$

$= E\{y(n)\,x(n-m)\} - m_x m_y$

Since sequence $x(n)$ and $y(n)$ are real, then $\Phi^*_{yx}(-m) = \Phi_{yx}(-m)$.

so $\Phi^*_{yx}(-m) = E\{x(n-m)\,y(n)\} - m_x m_y$ $\longrightarrow$

part (c) Cont.

show that $\Phi_{xy}(m) = \Phi^*_{yx}(-m)$.

$$\Phi_{xy}(m) = E\{X_n\, y^*_{n+m}\} = E\{y^*_{n+m}\, x_n\} = \left(E\{y_{n+m}\, x^*_n\}\right)^*$$

$$= \Phi^*_{yx}(-m)$$

show that $\gamma_{xy}(m) = \gamma^*_{yx}(-m)$

$$\gamma_{xy}(m) = E\{(X_n - m_x)(y^*_{n+m} - m^*_y)\}$$

$$= E\{(y^*_{n+m} - m^*_y)(X_n - m_x)\}$$

$$= \left(E\{(y_{n+m} - m_y)(x^*_n - m^*_x)\}\right)^*$$

$$= \gamma^*_{yx}(-m)$$

part (d) (9)

show that $|\Phi_{xy}(m)| \leq \sqrt{\Phi_{xx}(0)\,\Phi_{yy}(0)}$

$$\Phi_{xy}(m) = E\{x_n\, y^*_{n+m}\}$$
$$\Phi_{xx}(0) = E\{x_n^2\}$$
$$\Phi_{yy}(0) = E\{y_n^2\}.$$

we did this in class as follows:

$$0 \leq E\{(X_n + a\,y_{n+m})^2\} = E\{X_n^2 + a^2 y_{n+m}^2 + 2a X_n y_{n+m}\}$$

$$= E(X_n^2) + a^2 E(y_{n+m}^2) + 2a\,E(X_n\, y_{n+m})$$

$$= \Phi_{xx}(0) + a^2\,\Phi_{yy}(0) + 2a\,\Phi_{xy}(m) \quad \equiv Ax^2+Bx+C$$

this is a quadratic equation that is ≥ 0 always. hence can't have 2 real roots. i.e discriminant ≤ 0. i.e

where $A = \Phi_{yy}(0)$, $B = 2\Phi_{xy}(m)$, $C = \Phi_{xx}(0)$.

but discriminant is $B^2 - 4AC$

so $$4\Phi_{xy}^2(m) - 4\,\Phi_{yy}(0)\,\Phi_{xx}(0) \leq 0.$$

i.e

$$\Phi_{xy}^2(m) \leq \Phi_{yy}(0)\,\Phi_{xx}(0)$$

i.e $$\boxed{|\Phi_{xy}(m)| \leq \sqrt{\Phi_{yy}(0)\,\Phi_{xx}(0)}}$$

4.2.2 key solution

H.W. #3 Sol. 1/.

(1) a) Autocorrelation sequence : $\phi_{xx}(n,m)$ is defined by

$$\phi_{xx}(n,m) = E\{x_n x_m^*\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_n x_m^* \, P_{x_n x_m}(X_n, n, X_m, m)\, dX_n\, dX_m$$

b) A random process $\{x_n\}$ is a stationary process if its statistics are not affected by a shift in the time origin. i.e., x_n and x_m have the same statistics for all n and m

c) A stationary random process in the wide sense mean
(i) The mean is constant
(ii) the autocorrelation (2^{nd} order statistics) depend only on the time difference between the random variables

d) Time average of a random process $\{x_n\}$ is defined as

$$\langle x_n \rangle = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x_n$$

Ensemble average of a random process $\{x_n\}$ is defined as

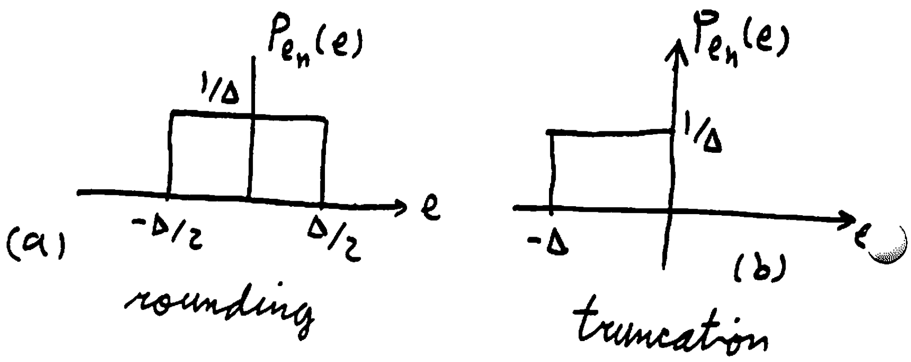
$$m_{x_n} = E\{x_n\} = \int_{-\infty}^{\infty} x\, P_{x_n}(x, n)\, dx$$

e) White noise is a random process in which all the random variables are independent. with zero mean

$$\phi_{xx}(m) = \sigma_x^2\, \delta(m)$$

f) A random process for which the time averages equal the ensemble averages is called an ergodic process.

② 8.3



Prob. distribution

a) Mean & variance, rounding

$$m_e = \int_{-\infty}^{\infty} e\, P_{e_n}(e)\, de = \int_{-\Delta/2}^{\Delta/2} e\, \frac{1}{\Delta}\, de = \frac{1}{\Delta}\, \frac{e^2}{2}\Bigg|_{-\Delta/2}^{\Delta/2} = 0$$

$$\sigma_e^2 = E\{e_n^2\} = \int_{-\infty}^{\infty} e^2 P_{e_n}(e)\, de = \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} e^2 de$$

$$= \frac{e^3}{3\Delta}\Bigg|_{-\Delta/2}^{\Delta/2} = \frac{1}{3\Delta}\, 2\, \frac{\Delta^3}{8} = \Delta^2/12$$

b) For truncation

$$m_e = \frac{1}{\Delta}\int_{-\Delta}^{0} e \, de = \frac{1}{\Delta}\,\frac{e^2}{2}\Big|_{-\Delta}^{0} = -\Delta/2$$

$$\underline{\sigma_e^2} = E\left\{\left(e_n+\frac{\Delta}{2}\right)^2\right\} = E\{e_n^2\} + \frac{\Delta^2}{4} + 2\frac{\Delta}{2}E\{e_n\}$$

$$= E\{e_n^2\} + \frac{\Delta^2}{4} - \frac{\Delta^2}{2} = \underline{E\{e_n^2\} - \frac{\Delta^2}{4}}$$

$$\sigma_e^2 = \frac{1}{\Delta}\int_{-\Delta}^{0} e^2 \, de - \frac{\Delta^2}{4} = \frac{1}{\Delta}\,\frac{e^3}{3}\Big|_{-\Delta}^{0} - \frac{\Delta^2}{4} = \frac{\Delta^2}{12}$$

③ 8.4 $e(n)$: white noise seq.
$s(n)$: uncorrelated with e_n

show $y(n) = s(n)\,e(n)$ is white ; i.e.

$$E\{y(n)\,y(n+m)\} = \underbrace{A}_{\text{const.}}\,\delta(m)$$

Sol.

$e(n)$ white $\Rightarrow E\{e(n)\,e(n+m)\} = \sigma_e^2\,\delta(m)$
uncorrelated $E\{e(n)\,y(m)\} = E\{e(n)\}\,E\{y(m)\}$

$$E\{y(n)\,y(n+m)\} = E\{s(n)\,e(n)\,s(n+m)\,e(n+m)\}$$

assume $s(n)$ is WSS or white noise

$$= E\{s(n)\,s(n+m)\,e(n)\,e(n+m)\}$$

$$= E\{s(n)\,s(n+m)\}\,E\{e(n)\,e(n+m)\}$$

$$= E\{s(n)\,s(n+m)\}\,\sigma_e^2\,\delta(m)$$

$$= \sigma_s^2\,\sigma_e^2\,\delta(m)$$

4/7

8.6 Consider the two real stationary random processes $\{x_n\}$ and $\{y_n\}$. with means m_x and m_y and variances σ_x^2 and σ_y^2. Show the following

(a) $\gamma_{xx}(m) = \phi_{xx}(m) - m_x^2$ & $\gamma_{xy}(m) = \phi_{xy}(m) - m_x \cdot m_y$

$$\gamma_{xx}(m) = E[(x_n - m_x)(x_{n+m} - m_x)]$$
$$= E[x_n x_{n+m}] - m_x E[x_{n+m}] - m_x E[x_n] + m_x m_x$$
$$= \phi_{xx}(m) - m_x m_x - m_x m_x + m_x m_x$$
$$= \phi_{xx}(m) - m_x^2$$

$$\gamma_{xy}(m) = E[(x_n - m_x)(y_{n+m} - m_y)]$$
$$= E[x_n y_{n+m}] - m_x m_y - m_y m_x + m_x m_y$$
$$= \phi_{xy}(m) - m_x m_y$$

(b) $\phi_{xx}(0)$ = mean square & $\gamma_{xx}(0) = \sigma_x^2$

$\phi_{xx}(m) = E[x_n x_{n+m}] =$

$\phi_{xx}(0) = E[x_n x_n]$ = mean square

$\gamma_{xx}(m) = E[(x_n - m_x)(x_{n+m} - m_x)]$

$\gamma_{xx}(0) = E[(x_n - m_x)^2] = \sigma_x^2$

(c) $\phi_{xx}(m) = \phi_{xx}(-m)$

$\phi_{xx}(-m) = (E[x_n x_{n-m}])$

let $n' = n - m$

$\phi_{xx}(-m) = (E[x_{n'+m} x_{n'}]) = E[x_{n'} x_{n'+m}]$

$= \phi_{xx}(m)$

$\gamma_{xx}(m) = \gamma_{xx}(-m)$

$$\gamma_{xx}(-m) = (E[(x_n - m_x)(x_{n-m} - m_x)])$$
$$= (E[(x_{n'+m} - m_x)(x_{n'} - m_x)])$$
$$= E[(x_{n'} - m_x)(x_{n'+m} - m_x)]$$
$$= \gamma_{xx}(m)$$

5/7

$\underline{\phi_{xy}(m) = \phi_{yx}(-m)}$

$$\begin{aligned}\phi_{yx}(-m) &= (E[(y_n - m_y)(x_{n-m} - m_x)]) \\ &= (E[(y_{n'+m} - m_y)(x_{n'} - m_x)]) \\ &= E[(x_{n'} - m_x)(y_{n'+m} - m_y)] \\ &= \phi_{xy}(m)\end{aligned}$$

$\underline{\gamma_{xy}(m) = \gamma_{yx}(-m)}$

$$\begin{aligned}\gamma_{yx}(-m) &= (E[(y_n - m_y)(x_{y-m} - m_x)]) \\ &= (E[(y_{n'+m} - m_y)(x_{n'} - m_x)]) \\ &= E[(x_{n'} - m_x)(y_{n'+m} - m_y)] \\ &= \gamma_{xy}(m).\end{aligned}$$

(d) Consider the inequality

$$E\left\{\left(\frac{x_n}{(E[x_n^2])^{\frac{1}{2}}} - \frac{y_{n+m}}{(E[y^2_{n+m}])^{\frac{1}{2}}}\right)^2\right\} \geq 0$$

This is true since the quantity inside the brackets is >0 for all m and n.

Now

$$E\left[\frac{x_n^2}{E[x_n^2]}\right] + E\left[\frac{y^2_{n+m}}{E[y^2_{n+m}]}\right] - 2\frac{E[x_n y_{n+m}]}{(E[x_n^2])^{\frac{1}{2}}(E[y^2_{n+m}])^{\frac{1}{2}}} \geq 0$$

This can be written as

$$\frac{\phi_{xx}(0)}{\phi_{xx}(0)} + \frac{\phi_{yy}(0)}{\phi_{yy}(0)} - \frac{2\ \phi_{xy}(m)}{\phi_{xx}^{1/2}(0)\ \phi_{yy}^{1/2}(0)} \geq 0$$

$$\frac{\phi_{xy}(m)}{\phi_{xx}^{1/2}(0)\ \phi_{yy}^{1/2}(0)} \leq 1 \Rightarrow \boxed{[\phi_{xx}(0)\ \phi_{yy}(0)]^{1/2} \geq \phi_{xy}(m)}$$

Now if we replace x_n by $(x_n - m_x)$ and y_{n+m} by $(y_{n+m} - m_y)$ in the inequality we can manipulate it in the same way to get.

6/7

$$\boxed{[\gamma_{xx}(0)\,\gamma_{yy}(0)]^{1/2} \geq \gamma_{xy}(m)}$$

Letting $y_n = x_n$ we can specialize these inequalities to

$$\boxed{\begin{array}{c} \phi_{xx}(0) \geq \phi_{xx}(m) \\ \hline \gamma_{xx}(0) \geq \gamma_{xx}(m) \end{array}}$$

(e) Let $y_n = x_{n-n_0}$

$$\underline{\phi_{yy}(m)} = E[y_n\, y_{n+m}]$$
$$= E[x_{n-n_0}\, x_{n+m-n_0}]$$
$$= \underline{\phi_{xx}(m)}$$

Obviously $\underline{\gamma_{yy}(m) = \gamma_{xx}(m)}$ for the same reasons.

(f) Let $\gamma_{xx}(m) \longleftrightarrow \Gamma_{xx}(z)$

$\gamma_{xy}(m) \longleftrightarrow \Gamma_{xy}(z)$

$\Gamma_{xx}(z) \triangleq \sum_m \gamma_{xx}(m)\, z^{-m} \Rightarrow$ **(1)** $\gamma_{xx}(m) = \frac{1}{2\pi j}\oint_c \Gamma_{xx}(z)\, z^{m-1}\, dz$

$$\underline{\gamma_{xx}(0) =} \boxed{\sigma_x^2 = \frac{1}{2\pi j}\oint_c \Gamma_{xx}(z)\, z^{-1}\, dz}$$

(2) We have shown that $\gamma_{xx}(m) = \gamma_{xx}(-m)$

Therefore $\Gamma_{xx}(z) = \sum_{m=-\infty}^{\infty} \gamma_{xx}(m)\, z^{-m}$

$$\underline{\Gamma_{xx}(z^{-1})} = \sum_{m=-\infty}^{\infty} \gamma_{xx}(m)\, z^{m} = \sum_{p=-\infty}^{\infty} \gamma_{xx}(-p)\, z^{-p}$$

$$= \sum_{m=-\infty}^{\infty} \gamma_{xx}(m)\, z^{-m} = \underline{\Gamma_{xx}(z)}.$$

$p \to m \Rightarrow$

use $\gamma_{xy}(m) = \gamma^*_{yx}(-m)$

$$\Gamma_{xy}(z) = \sum_{m=-\infty}^{\infty} \gamma_{xy}(m)\, z^{-m} = \sum_{m=-\infty}^{\infty} \gamma^*_{yx}(-m)\, z^{-m}$$

$$= \left(\sum_{\ell=-\infty}^{\infty} \gamma_{yx}(\ell)\, z^{*\,\ell} \right)^*$$

$$= \left(\sum_{\ell=-\infty}^{\infty} \gamma_{yx}(\ell)\, \left(z^{*\,-1}\right)^{-\ell} \right)^* = \Gamma^*_{yx}(1/z^*$$

4.3 HW4, Some floating points computation

Local contents

4.3.1 my solution, First Problem

Looking at 2 floating points problems. The first to illustrate the problem when adding large number to small number. The second to illustrate the problem of subtracting 2 numbers close to each others in magnitude.

Investigate floating point errors generated by the following sum $\sum_{n=1}^{N}\frac{1}{n^2}$, compare the result to that due summation in forward and in reverse directions.

4.3.1.1 Analysis

When performing the sum in the forward direction, as in $1+\frac{1}{4}+\frac{1}{16}+\cdots+\frac{1}{N^2}$ we observe that very quickly into the sum, we will be adding relatively large quantity to a very small quantity. Adding a large number of a very small number leads to loss of digits as was discussed in last lecture. However, we adding in reverse order, as in $\frac{1}{N^2}+\frac{1}{(N-1)^2}+\frac{1}{(N-2)^2}+\cdots+1$, we see that we will be adding, each time, 2 quantities that are relatively close to each other in magnitude. This reduces floating point errors.

The following code and results generated confirms the above. $N=20,000$ was used. The computation was forced to be in single precision to be able to better illustrate the problem.

4.3.1.2 Computation and Results

This program prints the result of the sum in the forward direction

```
      PROGRAM main
      IMPLICIT NONE
      REAL :: s
      INTEGER :: n,MAX

      s  = 0.0;
      MAX = 20000;
      DO n = 1,MAX
         s = s + (1./n**2);
      END DO

      WRITE(*,1) s
1     format('sum = ', F8.6)
      END PROGRAM main


sum = 1.644725
```

now compare the above result with that when performing the sum in the reverse direction

```
PROGRAM main
      IMPLICIT NONE
      REAL :: s
      INTEGER :: n,MAX

      s  = 0.0;
      MAX = 20000;
      DO n = MAX,1,-1
         s = s + (1./n**2);
      END DO

      WRITE(*,1) s
1     format('sum = ', F8.6)
```

```
      END PROGRAM main

sum = 1.644884
```

The result from the reverse direction sum is the more accurate result. To proof this, we can use double precision and will see that the sum resulting from double precision agrees with the digits from the above result when using reverse direction sum

```
      PROGRAM main
      IMPLICIT NONE
      DOUBLE PRECISION :: s
      INTEGER :: n,MAX

      s  = 0.0;
      MAX = 20000;
      DO n = 1,MAX
         s = s + (1./n**2);
      END DO

      WRITE(*,1) s
1     format('sum = ', F18.16)
      END PROGRAM main

sum = 1.6448840680982091
```

4.3.1.3 Conclusion

In floating point arithmetic, avoid adding a large number to a very small number as this results in loss of digits of the small number. The above trick illustrate one way to accomplish this and still perform the required computation.

In the above, there was $1.644884 - 1.644725 = 1.59 \times 10^{-4}$ error in the sum when it was done in the forward direction as compared to the reverse direction (for $20,000$ steps).In relative term, this error is $\frac{1.644884-1.644725}{1.644884}100$ which is about 0.01% relative error.

4.3.2 my solution, second problem

Investigate the problem when subtracting 2 numbers which are close in magnitude. If a,b are 2 numbers close to each others, then instead of doing $a-b$ do the following $(a-b)\frac{(a+b)}{(a+b)} = \frac{a^2-b^2}{a+b}$. The following program attempts to illustrate this by comparing result from $a-b$ to that from $\frac{a^2-b^2}{a+b}$ for 2 numbers close to each others.

```
      PROGRAM main
      IMPLICIT NONE
      DOUBLE PRECISION :: a,b,diff

      a = 32.000008;
      b = 32.000002;
      diff = a-b;
      WRITE(*,1), diff
      diff = (a**2-b**2)/(a+b);
      WRITE(*,1), diff
1     format('diff = ', F18.16)
      END PROGRAM main

diff = 0.0000038146972656
diff = 0.0000038146972656
```

I need to look more into this as I am not getting the right 2 numbers to show this problem.

4.3.3 key solution

Sol. H.W. 4 EE 518A 1/6

9-6

$$y(n) = \alpha\, y(n-1) + x(n)$$

variables & coefficients : sign - & - magnitude

results of mult.'s : truncated

$$\Rightarrow \quad w(n) = Q\left[\alpha\, w(n-1)\right] + x(n)$$

$Q[\cdot]$: sign - & - mag. truncation.

possibility of a zero-input limit cycle

$$|w(n)| = |w(n-1)| \quad \forall\, n$$

show that if the ideal sys. is stable, then no zero - input limit cycle can exist. Is the same true for 2's complement truncation ?

Sol.

To have zero-input limit cycle

$$|w(n)| = |w(n-1)|$$

or

$$\left| Q\left[\alpha\, w(n-1)\right]\right| = |w(n-1)| \qquad (1)$$

stable sys. $\Rightarrow$ $|\alpha| < 1$

$$\Rightarrow \quad |\alpha\, w(n-1)| < |w(n-1)| \qquad (2)$$

a) For sign - & - mag. truncation.

$$\left.\begin{array}{ll} -2^{-b} < Q(x) - x \leq 0 & x \geq 0 \\ 0 \leq Q(x) - x < 2^{-b} & x < 0 \end{array}\right\} \text{add to notes}$$

2/6

$\Rightarrow |Q(x)| \leqslant |x|$ for $x \geqslant 0$ or $x < 0$

Let $x = \alpha\, W(n-1)$

$\Rightarrow |Q[\alpha W(n-1)]| \leqslant |\alpha W(n-1)|$ (3)

(3) & (2) $\Rightarrow |Q[\alpha W(n-1)]| \leqslant |\alpha W(n-1)| < |W(n-1)|$

Since (1) is not satisfied no zero input limit cycle is possible.

b) For $Q[\cdot]$ = two's complement

$-2^{-b} < Q(x) - x \leqslant 0 \quad \forall x$

If $\underline{x > 0}$ $\quad x \geqslant Q[x]$ or $|x| \geqslant |Q[x]|$ (4)

If $\underline{x < 0}$ $\quad |Q[x]| \geqslant |x|$ (5)

For $\alpha W(n-1) > 0$

$|Q[\alpha W(n-1)]| \underset{4}{\leqslant} |\alpha W(n-1)| \underset{2}{<} |W(n-1)|$

$\Rightarrow$ no limit cycle : (1) is not satisfied

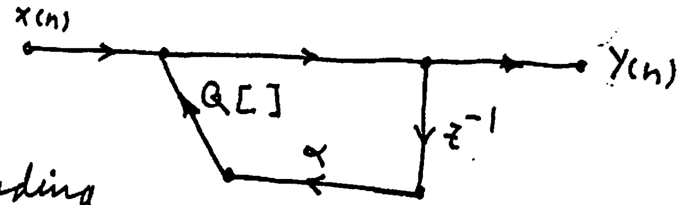
For $\alpha W(n-1) < 0$

$|\alpha W(n-1)| \leqslant |Q[\alpha W(n-1)]|$ by (5)

and $|\alpha W(n-1)| < |W(n-1)|$ by (2)

Possible that $|Q[\alpha W(n-1)]| = |W(n-1)|$ for $\alpha W(n-1) < 0$ $\Rightarrow$ limit cycle

9-7



Q[ ] : rounding

Fixed-pt. fractions, b bits

zero input $-\ y(-1) = A$ initial cond.

Dead band : $A \Rightarrow |Q[\alpha A]| = A$

a) dead band in terms of α and β

b) For $b=6$, $A = 1/16$ sketch $y(n)$ for $\alpha = \begin{cases} 15/16 \\ -15/16 \end{cases}$

c) For $b=6$, $A = 1/2$ sketch $y(n)$ for $\alpha = -15/16$

Sol.

$$y(n) = Q[\alpha\, y(n-1)] + x(n) \qquad (x(n)=0)$$

Rounding : $-\frac{2^{-b}}{2} < Q[\alpha W(n-1)] - \alpha W(n-1) \leq \frac{2^{-b}}{2}$

If filter is in the dead band

$$-\frac{2^{-b}}{2} < Q[\alpha A] - \alpha A \leq \frac{2^{-b}}{2}$$

or $|Q[\alpha A] - \alpha A| \leq \frac{2^{-b}}{2}$

In a limit cycle $|Q[\alpha A]| = A$

$\Rightarrow |Q[\alpha A]| - |\alpha A| \leq |Q[\alpha A] - \alpha A| \leq \frac{1}{2} 2^{-b}$

$\Rightarrow |A| - |\alpha||A| \leq \frac{1}{2} 2^{-b}$

$$\Rightarrow |A| \leq \frac{\frac{1}{2}\,2^{-b}}{1-|\alpha|}$$

b) $b = 6$ $\quad 2^{-b} = 1/64$ $\quad |\alpha| = 15/16$ $\quad 1-|\alpha| = 1/16$

$$|A| \leq \frac{\frac{1}{2}\cdot\frac{1}{64}}{\frac{1}{16}} = 1/8 \qquad \underline{\text{dead band}}$$

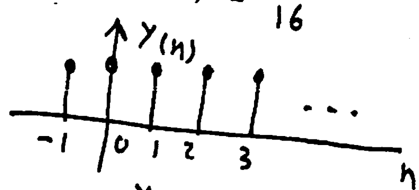
Thus for $A = 1/16$ the system starts immediately in the limit cycle.

$\alpha = \frac{15}{16}$ $\quad Y(n) = Q\left[\alpha\, Y(n-1)\right] = Q\left[\frac{15}{16}\cdot\frac{1}{16}\right] = Q\left[\frac{15}{256}\right] =$

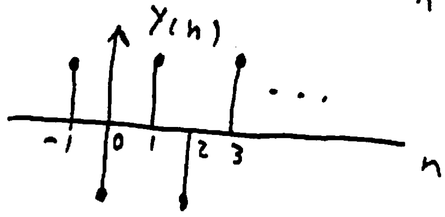
rounding!

$\alpha = \frac{-15}{16}$ $\quad Y(n) = Q\left[-\frac{15}{16}\cdot\frac{1}{16}\right] = \begin{cases} \frac{-1}{16} & n \text{ even} \\ \frac{1}{16} & n \text{ odd} \end{cases}$

$Y(-1) = \frac{1}{16}$



$\alpha = 15/16$

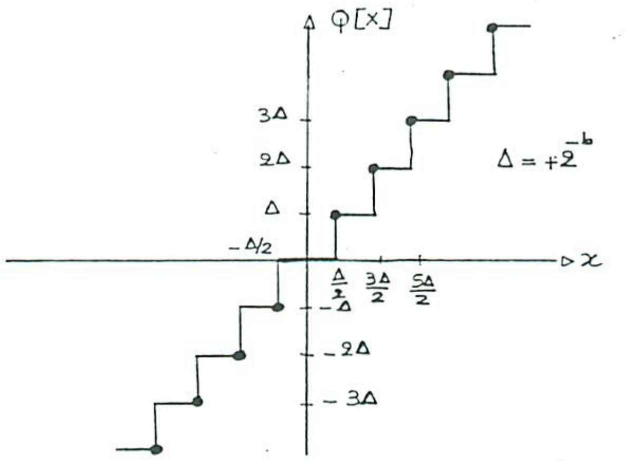


$\alpha = -15/16$

c) $b = 6$ $\quad A = 1/2$ $\quad \alpha = \frac{-15}{16}$ $\quad \Rightarrow$ same dead band

$$Y(n) = Q\left[\frac{-15}{16}\cdot Y(n-1)\right]$$

5/6



Thus we have: $w(0) = \varphi\left[-\frac{1}{2}\,\frac{15}{16}\right] = \varphi\left[-59\frac{\Delta}{2} - \frac{\Delta}{2}\right] = -30\Delta$

$w(1) = \varphi\left[\frac{15}{16}\cdot 30\Delta\right] = \varphi\left[56\frac{\Delta}{2} + \frac{1}{4}\frac{\Delta}{2}\right] = 28\Delta$

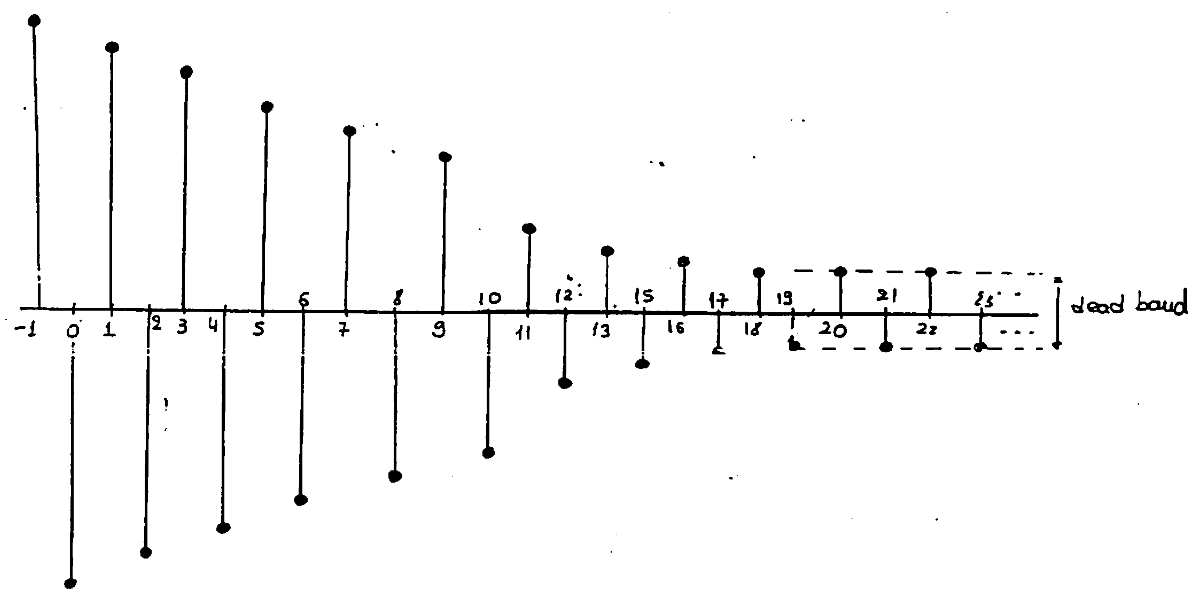
Hence we repeat the above procedure and we get:

$w(-1) = 32/64$
$w(0) = -30/64$
$w(1) = 28/64$
$w(2) = -26/64$
$w(3) = 24/64$ rounding up $\cancel{Q\left[\frac{-52.5}{128}\right]}$
$w(4) = -23/64$ → Round down! $Q\left[\frac{24\cdot 37}{64}\right]$!
$w(5) = 22/64$
$w(6) = -21/64$
$w(7) = 20/64$
$w(8) = -19/64$
$w(9) = 18/64$
$w(10) = -17/64$
$w(11) = 16/64$
$w(12) = -15/64$
$w(13) = 14/64$
$w(14) = -13/64$
$w(15) = 12/64$

8/8

$w(16) = -11/64$
$w(17) = 10/64$
$w(18) = -9/64$
$w(19) = 8/64$ ← rounding up
$w(20) = -8/64$
$w(21) = 8/64$
$w(22) = -8/64$

The output will be:



H.W. 5 sol. EE518A 1/2

11.1

$$C_{xx}(m) = \frac{1}{N}\sum_{n=0}^{N-|m|-1} x(n)\,x(n+m) \qquad |m| \leq N-1$$

show that

$$I_N(w) = \frac{1}{N}\left|X(e^{jw})\right|^2$$

$$I_N(w) = \sum_{m=-(N-1)}^{N-1} C_{xx}(m)\,e^{-jwm}$$

sol.

$$C_{xx}(m) = \frac{1}{N}\,x(n) * x(-n)$$

$$x(-n) \xrightarrow{F.T.} X(e^{-jw}) = X^*(e^{jw}) \quad \text{For } x(n) \text{ real}$$

$$\Rightarrow I_N(e^{jw}) = \frac{1}{N}X(e^{jw})\,X^*(e^{jw}) = \frac{1}{N}\left|X(e^{jw})\right|^2$$

or

$$I_N(w) \stackrel{\Delta}{=} \sum_{m=-(N-1)}^{N-1} C_{xx}(m)\,e^{-jwm}$$

$$= \sum_{m=-(N-1)}^{N-1}\left[\frac{1}{N}\sum_{n=0}^{N-|m|-1} x(n)\,x(n+m)\right]e^{-jwm}$$

$$= \frac{1}{N}\sum_{n=0}^{N-|m|-1} x(n)\sum_{m=-(N-1)}^{N-1} x(n+m)\,e^{-jwm}$$

$$= \frac{1}{N}\sum_{n=0}^{N-|m|-1} x(n)\sum_{\ell=n-(N-1)}^{n+(N-1)} x(\ell)\,e^{-jw\ell}\,e^{jwn}$$

$$= \frac{1}{N}\sum_{n=0}^{N-|m|-1} x(n)\,e^{jwn}\sum_{\ell=n-(N-1)}^{n+(N-1)} x(\ell)\,e^{-jw\ell}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x(n)\,e^{jwn}\sum_{\ell=0}^{N-1} x(\ell)\,e^{-jw\ell} \qquad \text{since } x(n)=0 \text{ for } n<0 \;\&\; n \geq N$$

2.1

$$I_N(\omega) = \frac{1}{N}\left[\sum_{n=0}^{N-1} x(n)\, e^{-j\omega n}\right]^* \sum_{\ell=0}^{N-1} x(\ell)\, e^{-j\omega \ell}$$

$$= \frac{1}{N}\left| X(e^{j\omega})\right|^2$$

$\underline{11.2}$ $$S_{xx}(\omega) = \sum_{m=-(M-1)}^{M-1} C_{xx}(m)\, w(m)\, e^{-j\omega m}$$

$w(m)$ of length $2M-1$

show that $$E\{S_{xx}(\omega)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} E\{I_N(\theta)\}\, W(e^{j(\omega-\theta)})\, d\theta$$

$$\begin{cases} W(m) = 0 & |m| \geq 2M \\ C_{xx}(m) = 0 & \text{for } |m| \geq M \end{cases}$$

Knowing these we can say

$$S_{xx}(\omega) = \sum_{m=-\infty}^{\infty} C_{xx}(m)\, W(m)\, e^{-j\omega m}$$

$$= \mathcal{F}\{C_{xx}(m)\, w(m)\}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathcal{F}\{C_{xx}(m)\}\, W(e^{j(\omega-\theta)})\, d\theta \quad \text{conv}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} I_N(\theta)\, W(e^{j(\omega-\theta)})\, d\theta$$

$$E\{S_{xx}(\omega)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} E\{I_N(\theta)\}\, W(e^{j(\omega-\theta)})\, d\theta$$

4.4 HW5

Local contents

4.4.1 Problem 11.1

1. Let $X(e^{j\omega})$ be the Fourier transform of a real finite-length sequence $x(n)$ that is zero outside the interval $0 \le n \le N-1$. The periodogram $I_N(\omega)$ is defined in Eq. (11.24) as the Fourier transform of the $2N-1$ point autocorrelation estimate

$$c_{xx}(m) = \frac{1}{N}\sum_{n=0}^{N-|m|-1} x(n)x(n+m) \qquad |m| \le N-1.$$

Show that the periodogram is related to the Fourier transform of the finite length sequence as follows:

$$I_N(\omega) = \frac{1}{N}|X(e^{j\omega})|^2.$$

Figure 4.2: the Problem statement

$$I_N(\omega) = \sum_{m=-(N-1)}^{N-1} c_{xx}(m)\, e^{-j\omega m}$$

$$\begin{aligned}
\left|X\left(e^{j\omega}\right)\right|^2 &= X\left(e^{j\omega}\right) X^*\left(e^{j\omega}\right)\\
&= \left(\sum_{m=0}^{N-1} x(m)\, e^{-j\omega m}\right)\left(\sum_{n=0}^{N-1} x(n)\, e^{-j\omega n}\right)^*\\
&= \left(\sum_{m=0}^{N-1} x(m)\, e^{-j\omega m}\right)\left(\sum_{n=0}^{N-1} x^*(n)\, e^{j\omega n}\right)\\
&= \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} x(m)\, x^*(n)\, e^{-j\omega m} e^{j\omega n}
\end{aligned}$$

But

$$e^{-j\omega m} e^{j\omega n} = e^{-j\omega(m-n)}$$

and

$$x(m)\, x^*(n) = x(m)\, x^*(m + (n-m))$$

So

$$\left|X\left(e^{j\omega}\right)\right|^2 = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} x(m)\, x^*(m+(n-m))\, e^{-j\omega(m-n)}$$

Let $n-m=\tau$ then above can be rewritten as

$$\left|X\left(e^{j\omega}\right)\right|^2 = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} x(m)\, x^*(m+\tau)\, e^{j\omega\tau}$$

When $n=0$,$m=-\tau$ and when $n=N-1$,$m=N-\tau-1$, hence the above becomes

$$\begin{aligned}
\left|X\left(e^{j\omega}\right)\right|^2 &= \sum_{m=0}^{N-1}\sum_{m=-\tau}^{N-\tau-1} x(m)\,x^*(m+\tau)\,e^{j\omega\tau} \\
&= \sum_{m=0}^{N-1}\left(\sum_{m=-\tau}^{-1} x(m)\,x^*(m+\tau)\,e^{j\omega\tau} + \sum_{m=0}^{N-|\tau|-1} x(m)\,x^*(m+\tau)\,e^{j\omega\tau}\right) \\
&= \sum_{m=0}^{N-1}\left(\sum_{m=-1}^{-\tau} x(m)\,x^*(m+\tau)\,e^{j\omega\tau} + N\ c_{xx}(m)\,e^{j\omega\tau}\right)
\end{aligned}$$

I made another attempt at the end,

4.4.2 Problem 11-2

2. The smoothed spectrum estimate $S_{xx}(\omega)$ is defined as

$$S_{xx}(\omega) = \sum_{m=-(M-1)}^{M-1} c_{xx}(m)w(m)e^{-j\omega m},$$

where $w(m)$ is a window sequence of length $2M-1$. Show that

$$E[S_{xx}(\omega)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} E[I_N(\theta)]\,W(e^{j(\omega-\theta)})\,d\theta,$$

where $W(e^{j\omega})$ is the Fourier transform of $w(n)$.

Figure 4.3: the Problem statement

We see that $S_{xx}(\omega)$ is the Fourier transform of $c_{xx}(m)\,w(m)$. i.e.

$$S_{xx}(\omega) = \mathcal{F}\left[c_{xx}(m)\,w(m)\right]$$

Where $\mathcal{F}$ is the Fourier transform operator. Using modulation property

$$S_{xx}(\omega) = \frac{1}{2\pi}\left(\mathcal{F}\left[c_{xx}(m)\right] \otimes \mathcal{F}\left[w(m)\right]\right)$$

But $I_N(\omega) = \mathcal{F}\left[c_{xx}(m)\right]$ and let $W(\omega) = \mathcal{F}\left[w(m)\right]$, then the above becomes

$$\begin{aligned}
S_{xx}(\omega) &= \frac{1}{2\pi}\left(I_N(\omega) \otimes W(\omega)\right) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} I_N(\theta)\,W(\omega-\theta)\,d\theta
\end{aligned}$$

Hence, taking expectation of LHS, and since only $I_N(\theta)$ is random, then the above becomes (after moving expectation inside the integral in the RHS)

$$E\left[S_{xx}(\omega)\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} E\left[I_N(\theta)\right]W(\omega-\theta)\,d\theta$$