

University Course

EGEE 518
Digital Signal Processing I

California State University, Fullerton
Fall 2008

My Class Notes
Nasser M. Abbasi

Fall 2008

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Chapter 1

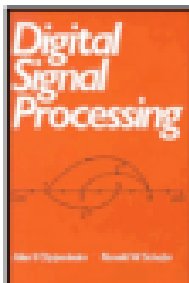
introduction

I took this course in Fall 2008 at CSUF to learn more about DSP.

This course was hard. The textbook was not too easy, The instructor Dr Shiva has tremendous experience in this subject, and he would explain some difficult things with examples on the board which helped quite a bit. The final exam was hard, it was 7 questions and I had no time to finish them all. It is a very useful course to take to learn about signal processing.

DIGITAL SIGNAL PROCESSING (CLOTH)

View: [Summary](#)



REQUIRED

Author: OPPENHEIM

Used : \$110.25 New : \$147.00

Quantity:

Instructor is professor Shiva, Mostaf, Dept Chair, EE, CSUF.



Chapter 2

Final project

final project

Chapter 3

Study notes

3.1 DSP notes

For fourier transform in mathematica, use these options

```
In[8]:= FourierTransform[1, t, s, FourierParameters -> {-1, 1}]  
Out[8]= DiracDelta[s]
```

From Wikipedia. Discrete convolution

Discrete convolution [\[edit\]](#)

For complex-valued functions f, g defined on the set of integers, the discrete convolution of f and g is given by:

$$\begin{aligned}(f * g)[n] &\stackrel{\text{def}}{=} \sum_{m=-\infty}^{\infty} f[m] \cdot g[n-m] \\ &= \sum_{m=-\infty}^{\infty} f[n-m] \cdot g[m]. \quad (\text{commutativity})\end{aligned}$$

Autocorrelaton

energy. Signals that "last forever" are treated instead as random processes, in which case different definitions are needed, based on expected values. For [wide-sense-stationary random processes](#), the autocorrelations are defined as

$$R_{ff}(\tau) = E[f(t)\bar{f}(t-\tau)]$$

$$R_{xx}(j) = E[x_n\bar{x}_{n-j}].$$

For processes that are not [stationary](#), these will also be functions of t , or n .

For processes that are also [ergodic](#), the expectation can be replaced by the limit of a time average. The autocorrelation of an ergodic process is sometimes defined as or equated to^[3]

$$R_{ff}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(t+\tau)\bar{f}(t) dt$$

$$R_{xx}(j) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} x_n\bar{x}_{n-j}.$$

These definitions have the advantage that they give sensible well-defined single-parameter results for periodic functions, even when those functions are not the output of stationary ergodic processes.

```

1 function nma_show_fourier
2
3 t=-4:.1:4;
4 N=4;
5 T=2;
6 plot(t,y(t,-N,N,T));
7
8
9 end
10
11 %-----
12 function v=c(k,T)
13 term=pi*k/2;
14 v=(1/T)*sin(term)/term;
15 end
16
17 %-----
18 function v=y(t,from,to,T)
19
20 coeff=zeros(to-from+1,1);
21 k=0;
22 for i=from:to
23     k=k+1;
24     coeff(k)=c(i,T);
25 end
26
27 v=zeros(length(t),1);
28 for i=1:length(t)
29     v(i)=0;
30     for k=from:to
31         v(i)=v(i)+coeff(k)*exp(sqrt(-1)*2*pi/T*k*t(i));
32     end

```


33 end
34 end

Chapter 4

HWs

4.1 HW2

Local contents

4.1.1	Problem 1	9
4.1.2	Problem 2	10
4.1.3	graded HW2	13

4.1.1 Problem 1

Compute an appropriate sampling rate and DFT size $N = 2^v$ to analyze a signal with no significant frequency content above 10kHz and with a minimum resolution of 100 Hz

4.1.1.1 Solution

From Nyquist sampling theory we obtain that sampling frequency is

$$f_s = 20000 \text{ Hz}$$

Now, the frequency resolution is given by

$$\Delta f = \frac{f_s}{N}$$

where N is the number of FFT samples. Now since the minimum Δf is 100 Hz then we write

$$\frac{f_s}{N} = \Delta f \geq 100$$

or

$$\frac{f_s}{N} \geq 100$$

Hence

$$\begin{aligned} N &\leq \frac{20,000}{100} \\ &\leq 200 \text{ samples} \end{aligned}$$

Therefore, we need the closest N below 200 which is power of 2, and hence

$$N = 128$$

4.1.2 Problem 2

sketch the locus of points obtained using Chirp Z Transform in the Z plane for $M = 8$, $W_0 = 2$, $\phi_0 = \frac{\pi}{16}$, $A_0 = 2$, $\theta_0 = \frac{\pi}{4}$

Answer:

Chirp Z transform is defined as

$$X(z_k) = \sum_{n=0}^{N-1} x[n] z_k^{-n} \quad k = 0, 1, \dots, M-1 \quad (1)$$

Where

$$z_k = AW^{-k}$$

and $A = A_0 e^{j\theta_0}$ and $W = W_0 e^{-j\phi_0}$

Hence

$$\begin{aligned} z_k &= (A_0 e^{j\theta_0}) (W_0 e^{-j\phi_0})^{-k} \\ &= \frac{A_0}{W_0^k} e^{j(\theta_0 + k\phi_0)} \end{aligned}$$

Hence

$$\begin{aligned} |z_k| &= \frac{A_0}{W_0^k} \\ &= \frac{2}{2^k} \end{aligned}$$

and

$$\begin{aligned} \text{phase of } z_k &= \theta_0 + k\phi_0 \\ &= \frac{\pi}{4} + k\frac{\pi}{16} \end{aligned}$$

Hence

k	$ z_k = \frac{2}{2^k}$	$\text{phase of } z_k = \frac{\pi}{4} + k\frac{\pi}{16}$	$\text{phase of } z_k \text{ in degrees}$
0	$\frac{2}{1} = 2$	$\frac{\pi}{4} + 0 \times \frac{\pi}{16} = \frac{\pi}{4}$	45
1	$\frac{2}{2} = 1$	$\frac{\pi}{4} + 1 \times \frac{\pi}{16} = \frac{5}{16}\pi$	56.25
2	$\frac{2}{4} = \frac{1}{2}$	$\frac{\pi}{4} + 2 \times \frac{\pi}{16} = \frac{3}{8}\pi$	67.5
3	$\frac{2}{8} = \frac{1}{4}$	$\frac{\pi}{4} + 3 \times \frac{\pi}{16} = \frac{7}{16}\pi$	78.75
4	$\frac{2}{16} = \frac{1}{8}$	$\frac{\pi}{4} + 4 \times \frac{\pi}{16} = \frac{1}{2}\pi$	90
5	$\frac{2}{32} = \frac{1}{16}$	$\frac{\pi}{4} + 5 \times \frac{\pi}{16} = \frac{9}{16}\pi$	101.25
6	$\frac{2}{64} = \frac{1}{32}$	$\frac{\pi}{4} + 6 \times \frac{\pi}{16} = \frac{5}{8}\pi$	112.5
7	$\frac{2}{128} = \frac{1}{64}$	$\frac{\pi}{4} + 7 \times \frac{\pi}{16} = \frac{11}{16}\pi$	123.75

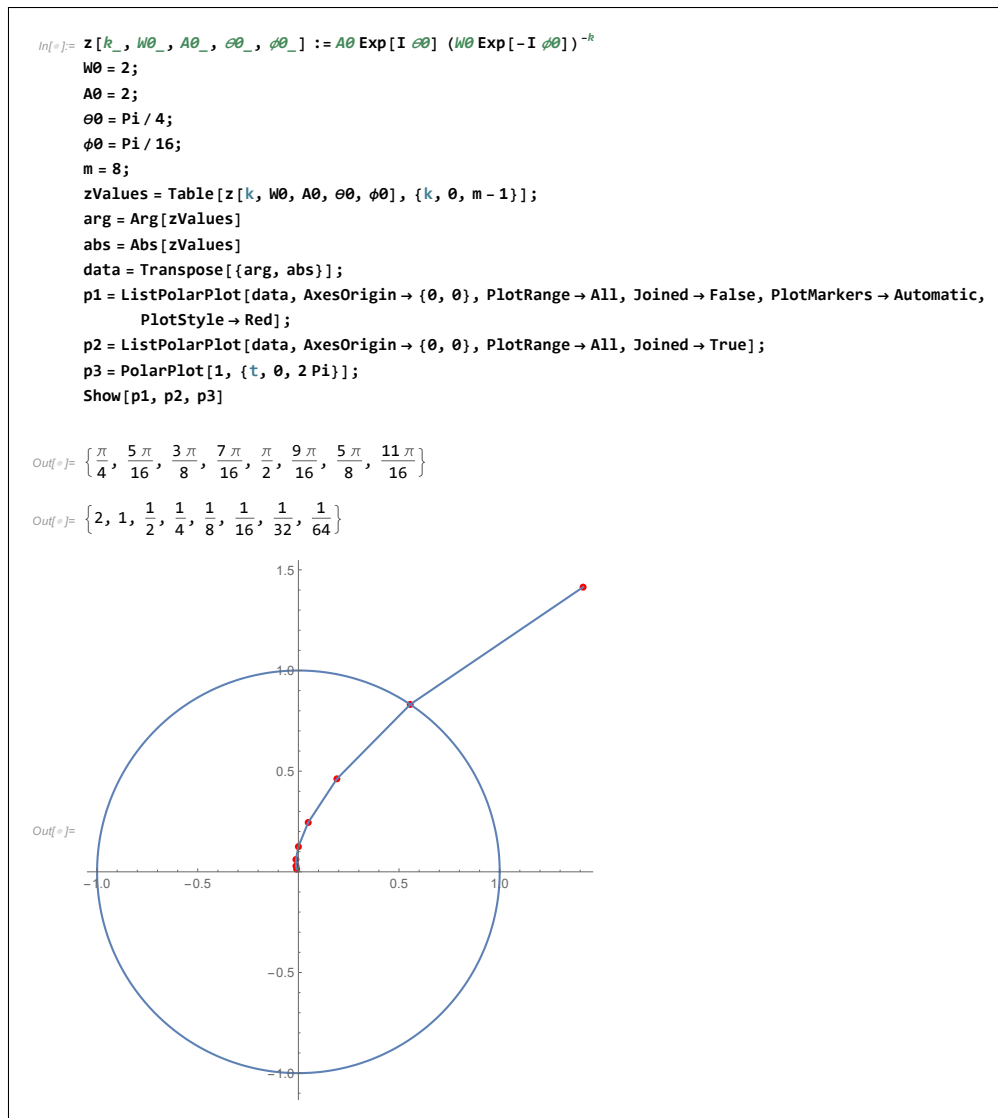


Figure 4.1: plot of the above contour

This is Mathematica notebook used to make plot of the Chirp Z transform contour. This is my graded HW2

4.1.3 graded HW2

HW2, EGEE 518. CSUF, Fall 2008

Nasser Abbasi

October 11, 2008

1 Problem 1

Compute an appropriate sampling rate and DFT size $N = 2^v$ to analyze a signal with no significant frequency content above 10kHz and with a minimum resolution of 100Hz

Solution

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or

$$\frac{f_s}{N} \geq 100$$

Hence

$$N \leq \frac{20,000}{100} \leq 200 \text{ samples}$$

Therefore, we need the closest N below 200 which is power of 2, and hence

$$N = 128$$

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2 Problem 2

sketch the locus of points obtained using Chirp Z Transform in the Z plane for $M = 8, W_0 = 2, \phi_0 = \frac{\pi}{16}, A_0 = 2, \theta_0 = \frac{\pi}{4}$

Answer:

Chirp Z transform is defined as

$$X(z_k) = \sum_{n=0}^{N-1} x[n] z_k^{-n} \quad k = 0, 1, \dots, M-1 \quad (1)$$

Where

$$z_k = AW^{-k}$$

and $A = A_0 e^{j\theta_0}$ and $W = W_0 e^{-j\phi_0}$

Hence

$$\begin{aligned} z_k &= (A_0 e^{j\theta_0}) (W_0 e^{-j\phi_0})^{-k} \\ &= \frac{A_0}{W_0^k} e^{j(\theta_0 + k\phi_0)} \end{aligned}$$

Hence

$$\begin{aligned} |z_k| &= \frac{A_0}{W_0^k} \\ &= \frac{2}{2^k} \end{aligned}$$

and

$$\begin{aligned} \text{phase of } z_k &= \theta_0 + k\phi_0 \\ &= \frac{\pi}{4} + k\frac{\pi}{16} \end{aligned}$$

Hence

k	$ z_k = \frac{2}{2^k}$	$\text{phase of } z_k = \frac{\pi}{4} + k\frac{\pi}{16}$	$\text{phase of } z_k \text{ in degrees}$
0	$\frac{2}{1} = 2$	$\frac{\pi}{4} + 0 \times \frac{\pi}{16} = \frac{\pi}{4}$	45
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Below is plot of the above contour

```

In[579]= z[k_, w0_, a0_, e0_, phi_] := a0 Exp[I e0] (w0 Exp[-I phi])^-k
w0 = 2;
a0 = 2;
e0 = Pi / 4;
phi = Pi / 16;
m = 8;
zValues = Table[z[k, w0, a0, e0, phi], {k, 0, m - 1}];
arg = Arg[zValues]
abs = Abs[zValues]
data = Transpose[{arg, abs}];
p1 = ListPolarPlot[data, AxesOrigin -> {0, 0},
  PlotRange -> All, Joined -> False,
  PlotMarkers -> {Automatic, Automatic}];
p2 = ListPolarPlot[data, AxesOrigin -> {0, 0},
  PlotRange -> All, Joined -> True];
p3 = PolarPlot[1, {t, 0, 2 Pi}];
Show[p1, p2, p3]

```

```

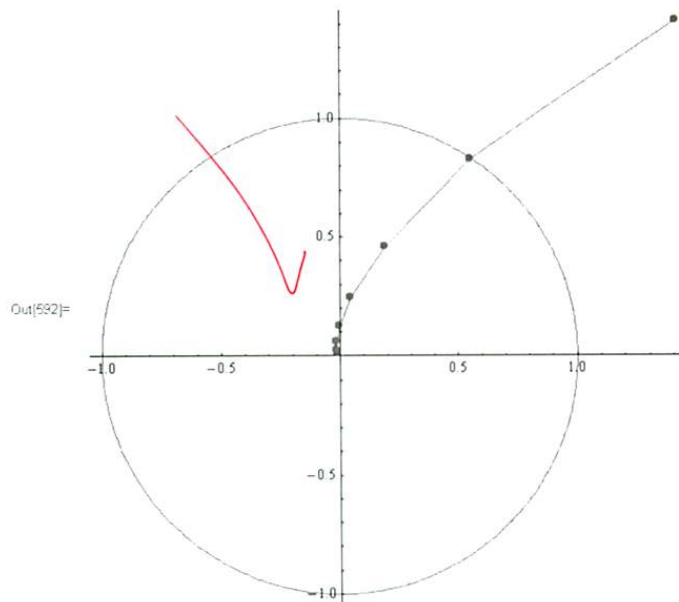
Out[586]= { $\frac{\pi}{4}$ ,  $\frac{5\pi}{16}$ ,  $\frac{3\pi}{8}$ ,  $\frac{7\pi}{16}$ ,  $\frac{\pi}{2}$ ,  $\frac{9\pi}{16}$ ,  $\frac{5\pi}{8}$ ,  $\frac{11\pi}{16}$ }

```

```

Out[587]= {2, 1,  $\frac{1}{2}$ ,  $\frac{1}{4}$ ,  $\frac{1}{8}$ ,  $\frac{1}{16}$ ,  $\frac{1}{32}$ ,  $\frac{1}{64}$ }

```



4.2 HW3

Local contents

4.2.1	my solution	18
4.2.2	key solution	27

4.2.1 my solution

①

Definitions

① auto correlation $R_{xx}(n, n+m)$: Measures the similarity of R.P. $X(t)$ at time n and $X(t)$ at later time $n+m$.

$$R_{xx}(n, n+m) = E\{X(n) X^*(n+m)\}$$

② stationary process.
This is a random process whose statistics do not change with shift in time origin.

③ wide sense stationary process:
This is a random process $X(t)$ which satisfies the following conditions:

1. its mean is constant. i.e. $E[X] = \text{constant}$.
2. auto correlation depends only on time interval 'm'.
i.e. $R_{xx}(n, n+m) = R_{xx}(m)$.

Notice that stationary process is WSS, but WSS is not necessarily stationary i.e. WSS

④ Time averages, Ensemble averages

Time averages is the average of the sample sequence.
while Ensemble average is statistical mean.

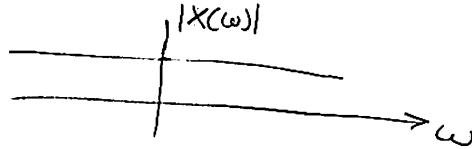
$X(t_n)$ is R.V.

t_n → ensemble average is $E[X(t_n)]$

⑤ White Noise:

this is a R.P. whose power spectral density is constant.
i.e. power contained in a frequency bandwidth B is the same regardless of where this bandwidth is centered.

"flat" spectrum
implies $X(t)$ is
white noise process.



The above is a description in the frequency domain.

In the time domain, $\Phi_{XX}(m) = \delta(m)$ i.e.

the autocorrelation is non-zero only if time interval is zero. i.e. $X(t)$ only correlates with itself at zero time delay. So all R.V. that belong to

a white noise process are uncorrelated with each other if time interval is non-zero.

⑥ Ergodic Process:

This is a R.P. when statistics taken from the time samples are the same as statistics taken from Ensembles.

For example, we say a process is Ergodic in the mean, then

$$E\{X(t)\} = \langle X(t) \rangle$$

↓
Statistical sample.
expected value of
R.V.

↓
time averaging.
mean of a sample (or
time series)

The above equality is in the limit, i.e. as the time series length increases, and the statistical mean is when the number of time series increases as well.

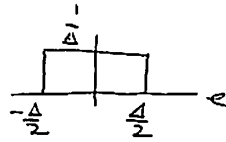
#3

③

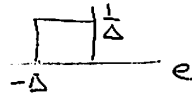
$$y(n) = Q[x(n)] = x(n) + e(n) \quad \rightarrow \text{quantization Error}$$

$e(n)$ is white Noise.

Pdf for rounding is uniform



Pdf for truncation is



- a) Find mean and Variance due to rounding
 b) " " " " " " truncation.

Answer

$$a) m_e = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e \cdot f(e) de = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \frac{e}{\Delta} de = \frac{1}{\Delta} \left(\frac{e^2}{2} \right)_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}}$$

$$= \frac{1}{2\Delta} \left[\left(\frac{\Delta}{2} \right)^2 - \left(-\frac{\Delta}{2} \right)^2 \right] = \frac{1}{2\Delta} (0) = \boxed{0}$$

$$E[e^2] = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e^2 f(e) de = \frac{1}{\Delta} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e^2 de = \frac{1}{\Delta} \left(\frac{e^3}{3} \right)_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}}$$

$$= \frac{1}{3\Delta} \left[\left(\frac{\Delta}{2} \right)^3 - \left(-\frac{\Delta}{2} \right)^3 \right] = \frac{1}{3\Delta} \left[\frac{\Delta^3}{8} + \frac{\Delta^3}{8} \right] = \frac{1}{3\Delta} \left[\frac{\Delta^3}{4} \right]$$

$$= \boxed{\frac{\Delta^2}{12}}$$

$$\text{so } \sigma^2 = E[e^2] - (E[e])^2 = \frac{\Delta^2}{12} - 0^2 = \boxed{\frac{\Delta^2}{12}}$$

$$b) m_e = \int_{-\Delta}^0 e f(e) de = \int_{-\Delta}^0 e \frac{1}{\Delta} de = \frac{1}{\Delta} \left(\frac{e^2}{2} \right)_{-\Delta}^0 = \frac{1}{2\Delta} (0^2 - (-\Delta)^2)$$

$$= \frac{1}{2\Delta} (0 - \Delta^2) = \boxed{-\frac{\Delta}{2}}$$

$$E[e^2] = \int_{-\Delta}^0 e^2 f(e) de = \int_{-\Delta}^0 e^2 \frac{1}{\Delta} de = \frac{1}{\Delta} \left[\frac{e^3}{3} \right]_{-\Delta}^0$$

$$= \frac{1}{3\Delta} [0^3 - (-\Delta)^3] = \boxed{\frac{\Delta^2}{3}}$$

$$\text{so } \sigma^2 = E[e^2] - (E[e])^2 = \frac{\Delta^2}{3} - \left(-\frac{\Delta}{2} \right)^2 = \frac{\Delta^2}{3} - \frac{\Delta^2}{4} = \frac{4\Delta^2 - 3\Delta^2}{12} = \boxed{\frac{\Delta^2}{12}}$$

4 let $e(n)$ white noise sequence. Let $s(n)$ uncorrelated sequence to $e(n)$. Show that $y(n) = s(n)e(n)$ is white. i.e. $E[y(n)y(n+m)] = A\delta(m)$.

Answer

$$\begin{aligned} E[y(n)y(n+m)] &= E[s(n)e(n)s(n+m)e(n+m)] \\ &= E[s(n)s(n+m)e(n)e(n+m)] \end{aligned}$$

Since $e(n)$ and $s(n)$ are uncorrelated, hence independent, then we can write the above as

$$= E[s(n)s(n+m)] E[e(n)e(n+m)]$$

but $e(n)$ is white. hence $E[e(n)e(n+m)] = \delta(m)$ by definition of white signal.

$$\text{hence } \phi_{yy}(n,m) = E[s(n)s(n+m)] \delta(m).$$

$$\text{Now, when } m=0, \phi_{yy}(n,m) = E[s(n)s(n)] \cdot 1.$$

Since $s(n)$ is uncorrelated with white noise, then $m_s=0$ since $s(n)$ is also white.

$$\begin{aligned} \text{hence } E[s^2(n)] &= \text{Total average power in } s(n) \\ &= A \text{ some constant.} \end{aligned}$$

$$\text{hence when } m=0, \phi_{yy}(n,m) = A$$

$$\begin{aligned} \text{when } m \neq 0 \phi_{yy}(n,m) &= E[s(n)s(n+m)] \cdot 0 \\ &= 0 \end{aligned}$$

$$\text{Therefore } \boxed{\phi_{yy}(n,m) = A\delta(m)}$$

Since $\phi_{yy}(n,m)$ is function of only m , it is white signal.

#6 Consider 2 real stationary random processes $\{x_n\}$ and $\{y_n\}$, with mean m_x, m_y , and variance σ_x^2, σ_y^2 .

(a) $\gamma_{xx}(m)$. This is autocovariance .

$$\begin{aligned}\gamma_{xx}(m) &= E\{(x(n) - m_x)(x^*(n+m) - m_x^*)\} \\ &= E\{x(n)x^*(n+m) - m_x x(n) - m_x x^*(n+m) + m_x^2\} \\ &= E\{x(n)x^*(n+m)\} - m_x E\{x(n)\} - m_x E\{x^*(n+m)\} \\ &\quad + m_x^2 \\ &= \phi_{xx}(n, n+m) - m_x^2 - m_x E\{x^*(n+m)\} + m_x^2 \\ &= \phi_{xx}(n, n+m) - m_x E\{x^*(n+m)\}.\end{aligned}$$

but $\{x_n\}$ is stationary, so its statistics do not change with shift of time origin. hence $E\{x^*(n+m)\} = E\{x^*(n)\} = m_x$.

so above becomes

$$\gamma_{xx}(m) = \phi_{xx}(n, n+m) - m_x^2.$$

but $\phi_{xx}(n, n+m) = \phi_{xx}(m)$ since stationary hence

$$\boxed{\gamma_{xx}(m) = \phi_{xx}(m) - m_x^2}$$

$$\begin{aligned}\gamma_{xy}(m) &= E[(x(n) - m_x)(y^*(n+m) - m_y^*)] \\ &= E[x(n)y^*(n+m) - m_y^* x(n) - m_x y^*(n+m) + m_y^* m_x] \\ &= E\{x(n)y^*(n+m)\} - m_y^* E\{x(n)\} - m_x E\{y^*(n+m)\} + m_y^* m_x\end{aligned}$$

but due to stationarity, $E\{y^*(n+m)\} = m_y$. so above becomes

$$\begin{aligned}&= E\{x(n)y^*(n+m)\} - m_y m_x - m_x m_y + m_y m_x \\ &= E\{x(n)y^*(n+m)\} - m_y m_x.\end{aligned}$$

but $E\{x(n)y^*(n+m)\} = \phi_{xy}(m)$ since stationarity

$$\Rightarrow \boxed{\gamma_{xy}(m) = \phi_{xy}(m) - m_y m_x} \rightarrow$$

$$(b) \Phi_{XX}(0) = E\{X(n)X^*(n+m)\}$$

but $m=0$. hence

$$\Phi_{XX}(0) = E\{X(n)X^*(n)\} = E\{X^2(n)\}.$$

= mean square.

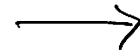
$$\gamma_{XX}(0) = E\{(X(n) - m_X)(X^*(n+m) - m_X^*)\}$$

but $m=0$ so

$$\begin{aligned} \gamma_{XX}(0) &= E\{(X(n) - m_X)(X^*(n) - m_X^*)\} \\ &= E\{X^2(n) - X(n)m_X - m_X X^*(n) + m_X^2\} \\ &= E\{X^2(n)\} - m_X E\{X(n)\} - m_X E\{X^*(n)\} + m_X^2 \\ &= E\{X^2(n)\} - m_X^2 - m_X^2 + m_X^2 \\ &= E\{X^2(n)\} - m_X^2 \end{aligned}$$

but this is the definition of σ_X^2 . hence

$$\boxed{\gamma_{XX}(0) = \sigma_X^2}$$



$$\begin{aligned} \text{(c)} \quad \Phi_{XX}(m) &= E \{ x(n) x^*(n+m) \} = E \{ x_{n+m}^* x_n \} = \left(E \{ x_{n+m} x_n^* \} \right)^* \textcircled{7} \\ &= \Phi_{XX}^*(-m) \end{aligned}$$

if process is real, then $\Phi_{XX}^*(-m) = \Phi_{XX}(-m)$.

$$\text{i.e. } \Phi_{XX}(m) = \Phi_{XX}(-m)$$

$$\begin{aligned} \gamma_{XX}(m) &= E \{ (x(n) - m_x) (x^*(n+m) - m_x^*) \} \\ &= \Phi_{XX}(m) - m_x m_x^* \quad (\text{from part (a)}) \quad \textcircled{1} \\ &= \Phi_{XX}^*(-m) - m_x m_x^* \quad (\text{using result above}) \\ &= \left(E \{ x_{n+m} x_n^* \} \right)^* - m_x m_x^* \\ &= E \{ x_{n+m}^* x_n \} - m_x m_x^* \\ &= \left(E \{ x_{n+m} x_n^* \} - m_x^* m_x \right)^* \\ &= \gamma_{XX}^*(-m) \end{aligned}$$

if Real process, then $\gamma_{XX}^*(-m) = \gamma_{XX}(-m) \Rightarrow \boxed{\gamma_{XX}(m) = \gamma_{XX}(-m)}$

~~$$\begin{aligned} \Phi_{XY}(m) &= E \{ (x(n) - m_x) (y(n+m) - m_y) \} \\ &= E \{ x(n) y(n+m) - m_y x(n) - m_x y(n+m) + m_x m_y \} \\ &= E \{ x(n) y(n+m) \} - m_y m_x - m_x m_y + m_x m_y \\ &= E \{ x(n) y(n+m) \} - m_y m_x \end{aligned}$$~~

~~$$\begin{aligned} \text{But } \Phi_{YX}(-m) &= E \{ (y(n) - m_y) (x(n-m) - m_x) \} \\ &= E \{ y(n) x(n-m) \} - m_x m_y - m_y m_x + m_y m_x \\ &= E \{ y(n) x(n-m) \} - m_x m_y \end{aligned}$$~~

Since sequences $x(n)$ and $y(n)$ are real, then $\Phi_{YX}(-m) = \Phi_{YX}^*(-m)$.

~~$$\text{So } \Phi_{YX}^*(-m) = E \{ x(n-m) y(n) \} - m_x m_y \rightarrow$$~~

part (c) cont.

show that $\Phi_{xy}(m) = \Phi_{yx}^*(-m)$.

$$\begin{aligned} \Phi_{xy}(m) &= E\{X_n y_{n+m}^*\} = E\{y_{n+m}^* X_n\} = \left(E\{y_{n+m} X_n^*\}\right)^* \\ &= \Phi_{yx}^*(-m) \end{aligned}$$

show that $\gamma_{xy}(m) = \gamma_{yx}^*(-m)$

$$\begin{aligned} \gamma_{xy}(m) &= E\{(X_n - m_x)(y_{n+m}^* - m_y^*)\} \\ &= E\{(y_{n+m}^* - m_y^*)(X_n - m_x)\} \\ &= \left(E\{(y_{n+m} - m_y)(X_n^* - m_x^*)\}\right)^* \\ &= \gamma_{yx}^*(-m) \end{aligned}$$

part (d)

9

show that $|\Phi_{xy}(m)| \leq \sqrt{\Phi_{xx}(0) \Phi_{yy}(0)}$

$$\Phi_{xy}(m) = E\{X_n Y_{n+m}^*\}$$

$$\Phi_{xx}(0) = E\{X_n^2\}$$

$$\Phi_{yy}(0) = E\{Y_n^2\}$$

we did this in class as follows:

$$0 \leq E\{(X_n + a Y_{n+m})^2\} = E\{X_n^2 + a^2 Y_{n+m}^2 + 2a X_n Y_{n+m}\}$$

$$= E(X_n^2) + a^2 E(Y_{n+m}^2) + 2a E(X_n Y_{n+m})$$

$$= \Phi_{xx}(0) + a^2 \Phi_{yy}(0) + 2a \Phi_{xy}(m) \quad (= Ax^2 + Bx + C)$$

this is a quadratic equation that is ≥ 0 always.

hence can't have 2 real roots. i.e.

discriminant ≤ 0 . i.e.

where $A = \Phi_{yy}(0)$, $B = 2\Phi_{xy}(m)$, $C = \Phi_{xx}(0)$.

but discriminant is $B^2 - 4AC$

$$\text{so } 4\Phi_{xy}^2(m) - 4\Phi_{yy}(0)\Phi_{xx}(0) \leq 0.$$

i.e.

$$\Phi_{xy}^2(m) \leq \Phi_{yy}(0)\Phi_{xx}(0)$$

i.e.

$$|\Phi_{xy}(m)| \leq \sqrt{\Phi_{yy}(0)\Phi_{xx}(0)}$$

4.2.2 key solution

H.W. #3 Sol.

① a) Autocorrelation sequence: $\phi_{xx}(n, m)$ is defined by

$$\phi_{xx}(n, m) = E\{X_n X_m^*\} = \iint_{-\infty}^{\infty} X_n X_m^* P_{X_n X_m}(X_n, n, X_m, m) dX_n dX_m$$

b) A random process $\{X_n\}$ is a stationary process if its statistics are not affected by a shift in the time origin. i.e., X_n and X_m have the same statistics for all n and m

c) A stationary random process in the wide sense means

(i) The mean is constant

(ii) the autocorrelation (2nd order statistics) depend only on the time difference between the random variables

d) Time average of a random process $\{X_n\}$ is defined as

$$\langle X_n \rangle = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N X_n$$

Ensemble average of a random process $\{X_n\}$ is defined as

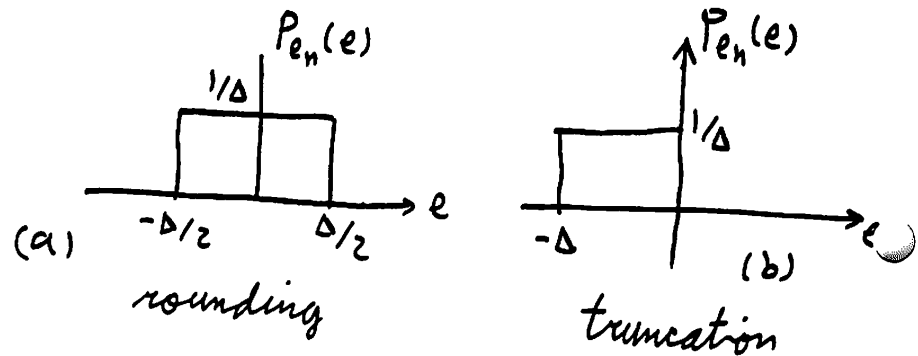
$$m_{X_n} = E\{X_n\} = \int_{-\infty}^{\infty} X P_{X_n}(X, n) dX$$

e) White noise is a random process in which all the random variables are independent. with zero mean

$$\Phi_{xx}(m) = \sigma_x^2 \delta(m)$$

f) A random process for which the time averages equal the ensemble averages is called an ergodic process.

(2) 8.3



Prob. distribution

a) Mean & variance, rounding

$$m_e = \int_{-\infty}^{\infty} e P_e(e) de = \int_{-\Delta/2}^{\Delta/2} e \frac{1}{\Delta} de = \frac{1}{\Delta} \frac{e^2}{2} \Big|_{-\Delta/2}^{\Delta/2} = 0$$

$$\sigma_e^2 = E\{e_n^2\} = \int_{-\infty}^{\infty} e^2 P_e(e) de = \frac{1}{\Delta} \int_{-\Delta/2}^{\Delta/2} e^2 de$$

$$= \frac{e^3}{3\Delta} \Big|_{-\Delta/2}^{\Delta/2} = \frac{1}{3\Delta} 2 \frac{\Delta^3}{8} = \frac{\Delta^2}{12}$$

b) For truncation

$$m_e = \frac{1}{\Delta} \int_{-\Delta}^0 e \, de = \frac{1}{\Delta} \left. \frac{e^2}{2} \right|_{-\Delta}^0 = -\frac{\Delta}{2}$$

$$\begin{aligned} \underline{\sigma_e^2} &= E \left\{ \left(e_n + \frac{\Delta}{2} \right)^2 \right\} = E \{ e_n^2 \} + \frac{\Delta^2}{4} + 2 \frac{\Delta}{2} E \{ e_n \} \\ &= E \{ e_n^2 \} + \frac{\Delta^2}{4} - \frac{\Delta^2}{2} = \underline{E \{ e_n^2 \} - \frac{\Delta^2}{4}} \end{aligned}$$

$$\sigma_e^2 = \frac{1}{\Delta} \int_{-\Delta}^0 e^2 \, de - \frac{\Delta^2}{4} = \frac{1}{\Delta} \left. \frac{e^3}{3} \right|_{-\Delta}^0 - \frac{\Delta^2}{4} = \frac{\Delta}{12}$$

③ 8.4 $e(n)$: white noise seq.
 $s(n)$: uncorrelated with $e(n)$

show $Y(n) = s(n)e(n)$ is white i.e.

$$E \{ Y(n) Y(n+m) \} = A \delta(m)$$

Sol.

$$\begin{cases} e(n) \text{ white} \Rightarrow E \{ e(n) e(n+m) \} = \sigma_e^2 \delta(m) \\ \text{uncorrelated} \quad E \{ e(n) Y(m) \} = E \{ e(n) \} E \{ Y(m) \} \end{cases}$$

$$E \{ Y(n) Y(n+m) \} = E \{ s(n) e(n) s(n+m) e(n+m) \}$$

$$= E \{ s(n) s(n+m) e(n) e(n+m) \}$$

$$= E \{ s(n) s(n+m) \} E \{ e(n) e(n+m) \}$$

$$= E \{ s(n) s(n+m) \} \sigma_e^2 \delta(m)$$

$$= \sigma_s^2 \sigma_e^2 \delta(m)$$

assume
 $s(n)$ is WSS
 or white noise

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8.6 Consider the two real stationary random processes $\{x_n\}$ and $\{y_n\}$, with means m_x and m_y and variances σ_x^2 and σ_y^2 . Show the following

$$(a) \delta_{xx}(m) = \phi_{xx}(m) - m_x^2 \quad \& \quad \delta_{xy}(m) = \phi_{xy}(m) - m_x m_y$$

$$\begin{aligned} \delta_{xx}(m) &= E[(x_n - m_x)(x_{n+m} - m_x)] \\ &= E[x_n x_{n+m}] - m_x E[x_{n+m}] - m_x E[x_n] + m_x m_x \\ &= \phi_{xx}(m) - m_x m_x - m_x m_x + m_x m_x \\ &= \phi_{xx}(m) - m_x^2 \end{aligned}$$

$$\begin{aligned} \delta_{xy}(m) &= E[(x_n - m_x)(y_{n+m} - m_y)] \\ &= E[x_n y_{n+m}] - m_x m_y - m_y m_x + m_x m_y \\ &= \phi_{xy}(m) - m_x m_y \end{aligned}$$

$$(b) \phi_{xx}(0) = \text{mean square} \quad \& \quad \delta_{xx}(0) = \sigma_x^2$$

$$\phi_{xx}(m) = E[x_n x_{n+m}] =$$

$$\phi_{xx}(0) = E[x_n x_n] = \text{mean square}$$

$$\delta_{xx}(m) = E[(x_n - m_x)(x_{n+m} - m_x)]$$

$$\delta_{xx}(0) = E[(x_n - m_x)^2] = \sigma_x^2$$

$$(c) \phi_{xx}(m) = \phi_{xx}(-m)$$

$$\phi_{xx}(-m) = (E[x_n x_{n-m}])$$

$$\text{let } n' = n - m$$

$$\begin{aligned} \phi_{xx}(-m) &= (E[x_{n'+m} x_{n'}]) = E[x_{n'} x_{n'+m}] \\ &= \phi_{xx}(m) \end{aligned}$$

$$\delta_{xx}(m) = \delta_{xx}(-m)$$

$$\begin{aligned} \delta_{xx}(-m) &= (E[(x_n - m_x)(x_{n-m} - m_x)]) \\ &= (E[(x_{n'+m} - m_x)(x_{n'} - m_x)]) \\ &= E[(x_{n'} - m_x)(x_{n'+m} - m_x)] \\ &= \delta_{xx}(m) \end{aligned}$$

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$$\begin{aligned} \phi_{xy}(m) &= \phi_{yx}(-m) \\ \phi_{yx}(-m) &= (E[(y_n - m_y)(x_{n-m} - m_x)]) \\ &= (E[(y_{n'+m} - m_y)(x_{n'} - m_x)]) \\ &= E[(x_{n'} - m_x)(y_{n'+m} - m_y)] \\ &= \phi_{xy}(m) \end{aligned}$$

$$\begin{aligned} \delta_{xy}(m) &= \delta_{yx}(-m) \\ \delta_{yx}(-m) &= (E[(y_n - m_y)(x_{y,m} - m_x)]) \\ &= (E[(y_{n'+m} - m_y)(x_{n'} - m_x)]) \\ &= E[(x_{n'} - m_x)(y_{n'+m} - m_y)] \\ &= \delta_{xy}(m). \end{aligned}$$

(d) Consider the inequality

$$E \left\{ \left(\frac{x_n}{(E[x_n^2])^{1/2}} - \frac{y_{n+m}}{(E[y_{n+m}^2])^{1/2}} \right)^2 \right\} \geq 0$$

This is true since the quantity inside the brackets is > 0 for all m and n .

Now

$$E \left[\frac{x_n^2}{E[x_n^2]} \right] + E \left[\frac{y_{n+m}^2}{E[y_{n+m}^2]} \right] - 2 \frac{E[x_n y_{n+m}]}{(E[x_n^2])^{1/2} (E[y_{n+m}^2])^{1/2}} \geq 0$$

This can be written as

$$\frac{\phi_{xx}(0)}{\phi_{xx}(0)} + \frac{\phi_{yy}(0)}{\phi_{yy}(0)} - \frac{2 \phi_{xy}(m)}{\phi_{xx}^{1/2}(0) \phi_{yy}^{1/2}(0)} \geq 0$$

$$\frac{\phi_{xy}(m)}{\phi_{xx}^{1/2}(0) \phi_{yy}^{1/2}(0)} \leq 1 \Rightarrow$$

$$\boxed{[\phi_{xx}(0) \phi_{yy}(0)]^{1/2} \geq \phi_{xy}(m)}$$

Now if we replace x_n by $(x_n - m_x)$ and y_{n+m} by $(y_{n+m} - m_y)$ in the inequality we can manipulate it in the same way to get

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$$[\gamma_{xx}(0) \gamma_{yy}(0)]^{1/2} \geq \gamma_{xy}(m)$$

Letting $y_m = x_m$ we can specialize these inequalities to

$$\begin{array}{l} \phi_{xx}(0) \geq \phi_{xx}(m) \\ \gamma_{xx}(0) \geq \gamma_{xx}(m) \end{array}$$

(e) Let $y_m = x_{m-n_0}$

$$\begin{aligned} \phi_{yy}(m) &= E[y_m y_{m+m}] \\ &= E[x_{m-n_0} x_{m+m-n_0}] \\ &= \phi_{xx}(m) \end{aligned}$$

Obviously $\gamma_{yy}(m) = \gamma_{xx}(m)$ for the same reasons.

(f) Let $\gamma_{xx}(m) \longleftrightarrow \Gamma_{xx}(z)$
 $\gamma_{xy}(m) \longleftrightarrow \Gamma_{xy}(z)$

$$\Gamma_{xx}(z) \triangleq \sum_m \gamma_{xx}(m) z^{-m} \Rightarrow (1)$$

$$\gamma_{xx}(m) = \frac{1}{2\pi j} \oint_c \Gamma_{xx}(z) z^{m-1} dz$$

$$\gamma_{xx}(0) = \sigma_x^2 = \frac{1}{2\pi j} \oint_c \Gamma_{xx}(z) z^{-1} dz$$

(2) We have shown that $\gamma_{xx}(m) = \gamma_{xx}(-m)$

Therefore $\Gamma_{xx}(z) = \sum_{m=-\infty}^{\infty} \gamma_{xx}(m) z^{-m}$

$$\begin{aligned} \Gamma_{xx}(z^{-1}) &= \sum_{m=-\infty}^{\infty} \gamma_{xx}(m) z^m = \sum_{p=-\infty}^{\infty} \gamma_{xx}(-p) z^{-p} \\ &= \sum_{m=-\infty}^{\infty} \gamma_{xx}(m) z^{-m} = \Gamma_{xx}(z) \end{aligned}$$

$f \rightarrow m \Rightarrow$

use $\gamma_{xy}(m) = \gamma_{yx}^*(-m)$

$$\Gamma_{xy}(z) = \sum_{m=-\infty}^{\infty} \gamma_{xy}(m) z^{-m} = \sum_{m=-\infty}^{\infty} \gamma_{yx}^*(-m) z^{-m}$$

$$= \left(\sum_{l=-\infty}^{\infty} \gamma_{yx}(l) z^{*l} \right)^*$$

$$= \left(\sum_{l=-\infty}^{\infty} \gamma_{yx}(l) (z^{*-1})^{-l} \right)^* = \Gamma_{yx}^*(1/z^*)$$

4.3 HW4, Some floating points computation

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4.3.1 my solution, First Problem

Looking at 2 floating points problems. The first to illustrate the problem when adding large number to small number. The second to illustrate the problem of subtracting 2 numbers close to each others in magnitude.

Investigate floating point errors generated by the following sum $\sum_{n=1}^N \frac{1}{n^2}$, compare the result to that due summation in forward and in reverse directions.

4.3.1.1 Analysis

When performing the sum in the forward direction, as in $1 + \frac{1}{4} + \frac{1}{16} + \dots + \frac{1}{N^2}$ we observe that very quickly into the sum, we will be adding relatively large quantity to a very small quantity. Adding a large number of a very small number leads to loss of digits as was discussed in last lecture. However, we adding in reverse order, as in $\frac{1}{N^2} + \frac{1}{(N-1)^2} + \frac{1}{(N-2)^2} + \dots + 1$, we see that we will be adding, each time, 2 quantities that are relatively close to each other in magnitude. This reduces floating point errors.

The following code and results generated confirms the above. $N = 20,000$ was used. The computation was forced to be in single precision to be able to better illustrate the problem.

4.3.1.2 Computation and Results

This program prints the result of the sum in the forward direction

```

1  PROGRAM main
2  IMPLICIT NONE
3  REAL :: s
4  INTEGER :: n,MAX
5
6  s = 0.0;
7  MAX = 20000;
8  DO n = 1,MAX
9     s = s + (1./n**2);
10 END DO
11
12 WRITE(*,1) s
13 1 format('sum = ', F8.6)
14 END PROGRAM main
15
16
```

```
17 sum = 1.644725
```

now compare the above result with that when performing the sum in the reverse direction

```
1 PROGRAM main
2   IMPLICIT NONE
3   REAL :: s
4   INTEGER :: n,MAX
5
6   s = 0.0;
7   MAX = 20000;
8   DO n = MAX,1,-1
9     s = s + (1./n**2);
10  END DO
11
12  WRITE(*,1) s
13 1  format('sum = ', F8.6)
14  END PROGRAM main
15
16 sum = 1.644884
```

The result from the reverse direction sum is the more accurate result. To proof this, we can use double precision and will see that the sum resulting from double precision agrees with the digits from the above result when using reverse direction sum

```
1   PROGRAM main
2   IMPLICIT NONE
3   DOUBLE PRECISION :: s
4   INTEGER :: n,MAX
5
6   s = 0.0;
7   MAX = 20000;
8   DO n = 1,MAX
9     s = s + (1./n**2);
10  END DO
11
12  WRITE(*,1) s
13 1  format('sum = ', F18.16)
14  END PROGRAM main
15
16 sum = 1.6448840680982091
```

4.3.1.3 Conclusion

In floating point arithmetic, avoid adding a large number to a very small number as this results in loss of digits of the small number. The above trick illustrate one way to accomplish this and still perform the required computation.

In the above, there was $1.644884 - 1.644725 = 1.59 \times 10^{-4}$ error in the sum when it was done in the forward direction as compared to the reverse direction (for 20,000 steps). In relative term, this error is $\frac{1.644884 - 1.644725}{1.644884} 100$ which is about 0.01% relative error.

4.3.2 my solution, second problem

Investigate the problem when subtracting 2 numbers which are close in magnitude. If a, b are 2 numbers close to each others, then instead of doing $a - b$ do the following $(a - b) \frac{(a+b)}{(a+b)} = \frac{a^2 - b^2}{a+b}$. The following program attempts to illustrate this by comparing result from $a - b$ to that from $\frac{a^2 - b^2}{a+b}$ for 2 numbers close to each others.

```

1      PROGRAM main
2      IMPLICIT NONE
3      DOUBLE PRECISION :: a,b,diff
4
5      a = 32.000008;
6      b = 32.000002;
7      diff = a-b;
8      WRITE(*,1), diff
9      diff = (a**2-b**2)/(a+b);
10     WRITE(*,1), diff
11     1  format('diff = ', F18.16)
12     END PROGRAM main
13
14     diff = 0.0000038146972656
15     diff = 0.0000038146972656

```

I need to look more into this as I am not getting the right 2 numbers to show this problem.

4.3.3 key solution

Sol. H.W. 4 EE 518A 1/6

9-6

$$Y(n) = \alpha Y(n-1) + X(n)$$

variables & coefficients : sign - & - magnitude
 results of mult.'s : truncated

$$\Rightarrow W(n) = Q[\alpha W(n-1)] + X(n)$$

$Q[\cdot]$: sign - & - mag. truncation.

possibility of a zero-input limit cycle

$$|W(n)| = |W(n-1)| \quad \forall n$$

show that if the ideal sys. is stable, then no zero-input limit cycle can exist. Is the same true for 2's complement truncation?

Sol.

To have zero-input limit cycle

$$|W(n)| = |W(n-1)|$$

or $|Q[\alpha W(n-1)]| = |W(n-1)|$ (1)

stable sys. $\Rightarrow |\alpha| < 1$

$$\Rightarrow |\alpha W(n-1)| < |W(n-1)|$$
 (2)

a) For sign - & - mag. truncation.

$$\begin{aligned} -2^{-b} < Q(x) - x \leq 0 & \quad x \geq 0 \\ 0 \leq Q(x) - x < 2^{-b} & \quad x < 0 \end{aligned}$$

} add to notes



$Q[]$: rounding

Fixed-pt. fractions, b bits

zero input - $Y(-1) = A$ initial cond.

Dead band: $A \Rightarrow |Q[\alpha A]| = A$

a) dead band in terms of α and β

b) For $b=6$, $A=1/16$ sketch $Y(n)$ for $\alpha = \begin{cases} 15/16 \\ -15/16 \end{cases}$

c) For $b=6$, $A=1/2$ sketch $Y(n)$ for $\alpha = -15/16$

Sol.

$$Y(n) = Q[\alpha Y(n-1)] + X(n) \quad (X(n)=0)$$

$$\text{Rounding: } -\frac{2^{-b}}{2} < Q[\alpha W(n-1)] - \alpha W(n-1) \leq \frac{2^{-b}}{2}$$

If filter is in the dead band

$$-\frac{2^{-b}}{2} < Q[\alpha A] - \alpha A \leq \frac{2^{-b}}{2}$$

$$\text{or } |Q[\alpha A] - \alpha A| \leq \frac{2^{-b}}{2}$$

In a limit cycle $|Q[\alpha A]| = A$

$$\Rightarrow |Q[\alpha A]| - |\alpha A| \leq |Q[\alpha A] - \alpha A| \leq \frac{1}{2} 2^{-b}$$

$$\Rightarrow |A| - |\alpha| |A| \leq \frac{1}{2} 2^{-b}$$

$$\Rightarrow |A| \leq \frac{\frac{1}{2} 2^{-b}}{1-|\alpha|}$$

$$b) \quad b=6 \quad 2^{-b} = 1/64 \quad |\alpha| = 15/16 \quad 1-|\alpha| = 1/16$$

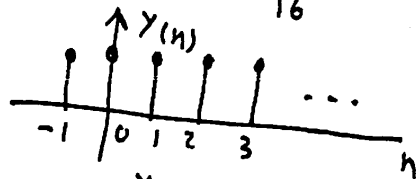
$$|A| \leq \frac{\frac{1}{2} \cdot \frac{1}{64}}{\frac{1}{16}} = 1/8 \quad \underline{\text{dead hand}}$$

Thus for $A = 1/16$ the system starts immediately in the limit cycle.

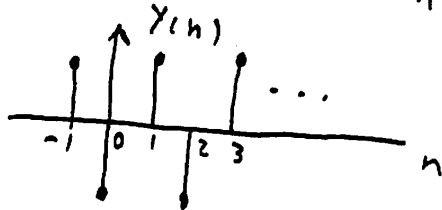
$$\alpha = \frac{15}{16} \quad Y(n) = Q[\alpha Y(n-1)] = Q\left[\frac{15}{16} \cdot \frac{1}{16}\right] = Q\left[\frac{15}{256}\right] = i$$

$$\alpha = -\frac{15}{16} \quad Y(n) = Q\left[-\frac{15}{16} \cdot \frac{1}{16}\right] = \begin{cases} -\frac{1}{16} & n \text{ even} \\ \frac{1}{16} & n \text{ odd} \end{cases} \quad \text{rounding!}$$

$$Y(-1) = \frac{1}{16}$$



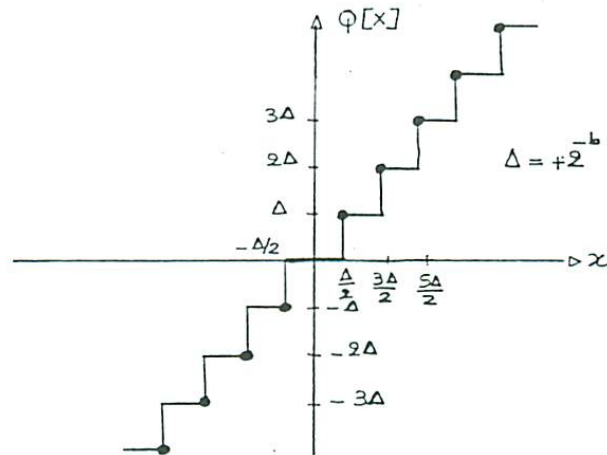
$$\alpha = 15/16$$



$$\alpha = -15/16$$

$$c) \quad b=6 \quad A=1/2 \quad \alpha = -\frac{15}{16} \quad \Rightarrow \text{same dead hand}$$

$$Y(n) = Q\left[-\frac{15}{16} \cdot Y(n-1)\right]$$



Thus we have: $w(0) = \varphi\left[-\frac{1}{2} \frac{15}{16}\right] = \varphi\left[-59 \frac{\Delta}{2} - \frac{\Delta}{2}\right] = -30\Delta$

$$w(1) = \varphi\left[\frac{15}{16} \cdot 30\Delta\right] = \varphi\left[56 \frac{\Delta}{2} + \frac{1}{4} \frac{\Delta}{2}\right] = 28\Delta$$

Hence we repeat the above procedure and we get:

$$w(-1) = 32/64$$

$$w(0) = -30/64$$

$$w(1) = 28/64$$

$$w(2) = -26/64$$

$$w(3) = 24/64$$

$$w(4) = -23/64$$

$$w(5) = 22/64$$

$$w(6) = -21/64$$

$$w(7) = 20/64$$

$$w(8) = -19/64$$

$$w(9) = 18/64$$

$$w(10) = -17/64$$

$$w(11) = 16/64$$

$$w(12) = -15/64$$

$$w(13) = 14/64$$

$$w(14) = -13/64$$

$$w(15) = 12/64$$

rounding up $\left\lceil \frac{-52.5}{128} \right\rceil$
 round down $\left\lfloor \frac{24.37}{64} \right\rfloor$!

6/6

(-)

$$w(16) = -11/64$$

$$w(17) = 10/64$$

$$w(18) = -9/64$$

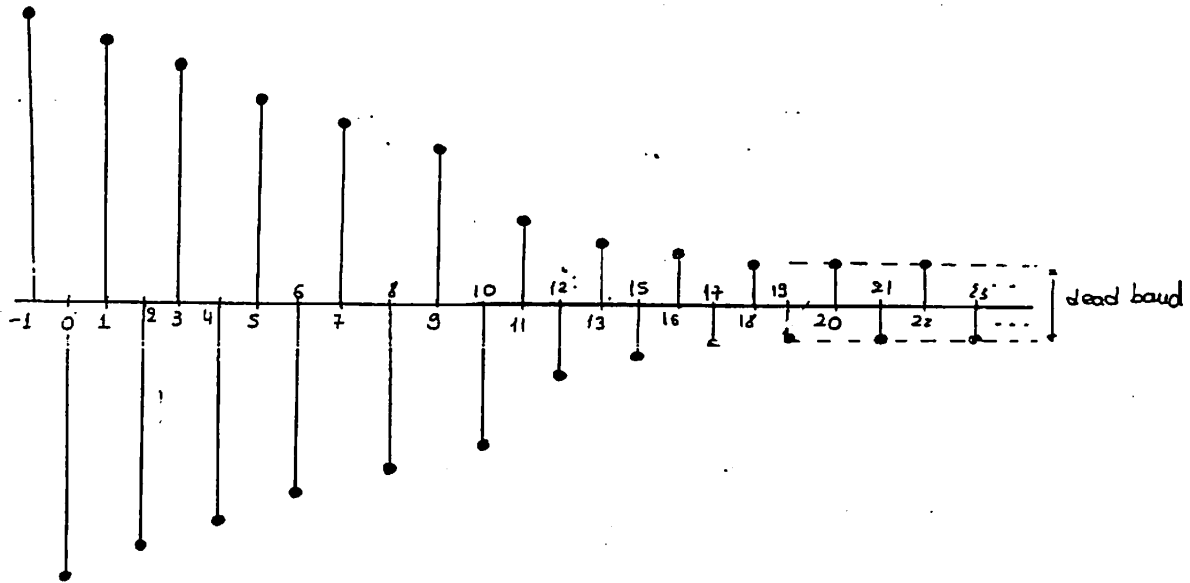
$$w(19) = 8/64 \quad \leftarrow \text{rounding up}$$

$$w(20) = -8/64$$

$$w(21) = 8/64$$

$$w(22) = -8/64$$

The output will be:



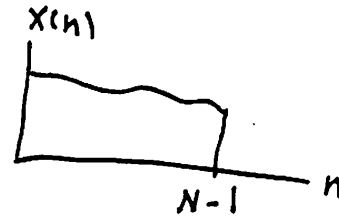
11.1H.W. 5 sol.EE518A

1/2

$$C_{xx}(m) = \frac{1}{N} \sum_{n=0}^{N-|m|-1} x(n)x(n+m) \quad |m| \leq N-1$$

show that

$$I_N(\omega) = \frac{1}{N} |\mathcal{X}(e^{j\omega})|^2$$



$$I_N(\omega) = \sum_{m=-(N-1)}^{N-1} C_{xx}(m) e^{-j\omega m}$$

Sol.

$$C_{xx}(m) = \frac{1}{N} x(n) * x(-n)$$

$$x(-n) \xrightarrow{\text{F.T.}} \mathcal{X}(e^{-j\omega}) = \mathcal{X}^*(e^{j\omega}) \quad \text{For } x(n) \text{ real}$$

$$\Rightarrow I_N(e^{j\omega}) = \frac{1}{N} \mathcal{X}(e^{j\omega}) \mathcal{X}^*(e^{j\omega}) = \frac{1}{N} |\mathcal{X}(e^{j\omega})|^2$$

or

$$I_N(\omega) \stackrel{A}{=} \sum_{m=-(N-1)}^{N-1} C_{xx}(m) e^{-j\omega m}$$

$$= \sum_{m=-(N-1)}^{N-1} \left[\frac{1}{N} \sum_{n=0}^{N-|m|-1} x(n)x(n+m) \right] e^{-j\omega m}$$

$$= \frac{1}{N} \sum_{n=0}^{N-|m|-1} x(n) \sum_{m=-(N-1)}^{N-1} x(n+m) e^{-j\omega m}$$

$$= \frac{1}{N} \sum_{n=0}^{N-|m|-1} x(n) \sum_{\ell=n-(N-1)}^{n+(N-1)} x(\ell) e^{-j\omega \ell} e^{j\omega n}$$

$$= \frac{1}{N} \sum_{n=0}^{N-|m|-1} x(n) e^{j\omega n} \sum_{\ell=n-(N-1)}^{n+(N-1)} x(\ell) e^{-j\omega \ell}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{j\omega n} \sum_{\ell=0}^{N-1} x(\ell) e^{-j\omega \ell} \quad \text{since } x(n)=0 \text{ for } n < 0 \text{ \& } n \geq N$$

$$I_N(\omega) = \frac{1}{N} \left[\sum_{n=0}^{N-1} x(n) e^{-j\omega n} \right]^* \sum_{l=0}^{N-1} x(l) e^{-j\omega l}$$

$$= \frac{1}{N} \left| \sum_{n=0}^{N-1} x(n) e^{-j\omega n} \right|^2$$

$$11.2 \quad S_{xx}(\omega) = \sum_{m=-(M-1)}^{M-1} C_{xx}(m) W(m) e^{-j\omega m}$$

$W(m)$ of length $2M-1$

show that
$$E \{ S_{xx}(\omega) \} = \frac{1}{2\pi} \int_{-\pi}^{\pi} E \{ I_N(\theta) \} W(e^{j(\omega-\theta)}) d\theta$$

$$\begin{cases} W(m) = 0 & |m| \geq 2M \\ C_{xx}(m) = 0 & \text{for } |m| \geq M \end{cases}$$

Assuming these we can say

$$S_{xx}(\omega) = \sum_{m=-\infty}^{\infty} C_{xx}(m) W(m) e^{-j\omega m}$$

$$= \mathcal{F} \{ C_{xx}(m) W(m) \}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{F} \{ C_{xx}(m) \} W(e^{j(\omega-\theta)}) d\theta \quad \text{conv}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} I_N(\theta) W(e^{j(\omega-\theta)}) d\theta$$

$$E \{ S_{xx}(\omega) \} = \frac{1}{2\pi} \int_{-\pi}^{\pi} E \{ I_N(\theta) \} W(e^{j(\omega-\theta)}) d\theta$$

4.4 HW5

Local contents

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4.4.1 Problem 11.1

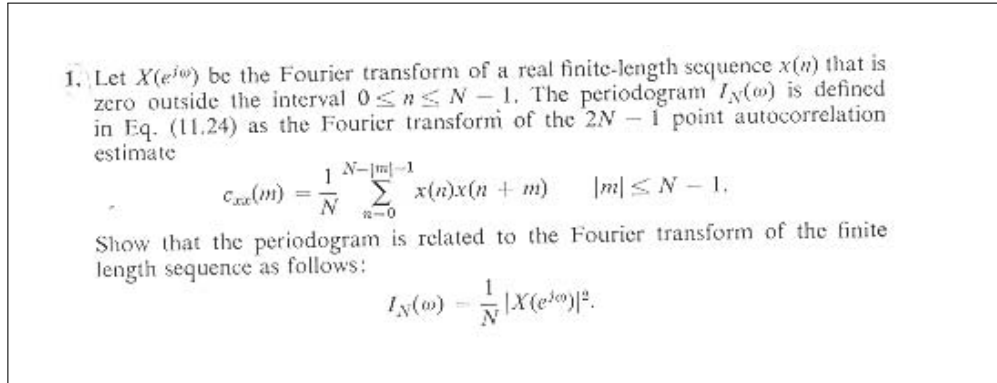


Figure 4.2: the Problem statement

$$I_N(\omega) = \sum_{m=-(N-1)}^{N-1} c_{xx}(m) e^{-j\omega m}$$

$$\begin{aligned} |X(e^{j\omega})|^2 &= X(e^{j\omega}) X^*(e^{j\omega}) \\ &= \left(\sum_{m=0}^{N-1} x(m) e^{-j\omega m} \right) \left(\sum_{n=0}^{N-1} x(n) e^{-j\omega n} \right)^* \\ &= \left(\sum_{m=0}^{N-1} x(m) e^{-j\omega m} \right) \left(\sum_{n=0}^{N-1} x^*(n) e^{j\omega n} \right) \\ &= \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} x(m) x^*(n) e^{-j\omega m} e^{j\omega n} \end{aligned}$$

But

$$e^{-j\omega m} e^{j\omega n} = e^{-j\omega(m-n)}$$

and

$$x(m) x^*(n) = x(m) x^*(m + (n - m))$$

So

$$|X(e^{j\omega})|^2 = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} x(m) x^*(m + (n - m)) e^{-j\omega(m-n)}$$

Let $n - m = \tau$ then above can be rewritten as

$$|X(e^{j\omega})|^2 = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} x(m) x^*(m + \tau) e^{j\omega\tau}$$

When $n = 0, m = -\tau$ and when $n = N - 1, m = N - \tau - 1$, hence the above becomes

$$\begin{aligned} |X(e^{j\omega})|^2 &= \sum_{m=0}^{N-1} \sum_{m=-\tau}^{N-1-\tau} x(m) x^*(m + \tau) e^{j\omega\tau} \\ &= \sum_{m=0}^{N-1} \left(\sum_{m=-\tau}^{-1} x(m) x^*(m + \tau) e^{j\omega\tau} + \sum_{m=0}^{N-|\tau|-1} x(m) x^*(m + \tau) e^{j\omega\tau} \right) \\ &= \sum_{m=0}^{N-1} \left(\sum_{m=-1}^{-\tau} x(m) x^*(m + \tau) e^{j\omega\tau} + N c_{xx}(m) e^{j\omega\tau} \right) \end{aligned}$$

I made another attempt at the end,

4.4.2 Problem 11-2

2. The smoothed spectrum estimate $S_{xx}(\omega)$ is defined as

$$S_{xx}(\omega) = \sum_{m=-(M-1)}^{M-1} c_{xx}(m) w(m) e^{-j\omega m},$$

where $w(m)$ is a window sequence of length $2M - 1$. Show that

$$E[S_{xx}(\omega)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} E[I_N(\theta)] W(e^{i(\omega-\theta)}) d\theta,$$

where $W(e^{j\omega})$ is the Fourier transform of $w(n)$.

Figure 4.3: the Problem statement

We see that $S_{xx}(\omega)$ is the Fourier transform of $c_{xx}(m) w(m)$. i.e.

$$S_{xx}(\omega) = F [c_{xx}(m) w(m)]$$

Where F is the Fourier transform operator. Using modulation property

$$S_{xx}(\omega) = \frac{1}{2\pi} (F [c_{xx}(m)] \otimes F [w(m)])$$

But $I_N(\omega) = F [c_{xx}(m)]$ and let $W(\omega) = F [w(m)]$, then the above becomes

$$\begin{aligned} S_{xx}(\omega) &= \frac{1}{2\pi} (I_N(\omega) \otimes W(\omega)) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} I_N(\theta) W(\omega - \theta) d\theta \end{aligned}$$

Hence, taking expectation of LHS, and since only $I_N(\theta)$ is random, then the above becomes (after moving expectation inside the integral in the RHS)

$$E [S_{xx}(\omega)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} E [I_N(\theta)] W(\omega - \theta) d\theta$$