**University Course**

# EGEE 443
# Electronic Communication systems

## California State University, Fullerton
## Fall 2008


My Class Notes

**Nasser M. Abbasi**


Fall 2008

# Contents

# Chapter 1

# Introduction

## 1.1 syllabus

CALIFORNIA STATE UNIVERSITY, FULLERTON
DEPARTMENT OF ELECTRICAL ENGINEERING

Fall 2008

EGEE 443 Electronic Communications (3)

COURSE DESCRIPTION:

electronics → prob. and statistics.

Prerequisites: EGEE 310 and EGEE 323.
Principles of amplitude, angular and pulse modulation, representative communication systems, the effect of noise on system performance.

MTW Th 1700 – 17:30
M TH 2015 – 20:45

INSTRUCTOR: K. HAMIDIAN

OFFICE: E-217

Advising
M TH 4pm – 5pm

TELEPHONE: 714–278-2884

FAX: 714–278-7162

E-217 office

OFFICE HOURS: MW: 1700– 17:30 and 2015-2045
TTH : 1700 – 17:30 and 2015-2045

Temporary.

PREREQUISITE TOPICS: Probability, Fourier Transforms, Linear Systems

TEXTBOOK: Intrduction to Analog & Digital Communications, S. Haykin and M.Moher, Wiley, 2007, 2th Edition

REFERENCES:

1) Introduction to Communication Systems, F. Stremler, Addison Wesley, 1982, 2nd edition

2) Digital and Analog Communication Systems, L. Couch, Prentice Hall, 2001, 6th edition.

3) Analog and Digital Communication Systems, M. Roden, Prentice Hall, 1996

get Class notes from book store.

2

COURSE OUTLINE

| WEEKS | TOPICS |
|---|---|
| 5.5 | Chapter 1. Introduction, Classification of Signals. Handout |
| | Chapter 2. Fourier Transform Review, Properties and Applications, Power and Energy Spectral Density. Band-pass Signals and Systems. Hilbert Transforms, Pre-Envelope, Quadrature Representation of Narrow Band Signals. Transmission of Signals Through Linear Systems. (apply Fourier Transform in communication) |
| | Chapter 8. Random Processes Stationary Processes. Ergodic Processes. Transmission of a Random Process Through a Linear-Time-Invariant Filter. Power Spectral Density. Gaussian Process Noise, Quadrature Representation of Narrowband Noise. Sine Wave Plus Narrowband Noise. |
| | MIDTERM 1 (75 MINUTES) (Complex envelope, Hilbert transform, Fourier Transform) |
| 6.0 | Chapters 2 and 9. Amplitude Modulation<br>Introduction, Amplitude Modulation (AM), Double Sidebanb-Suppresed Carrier (DSBSC), Single Sideband (SSB), Vestigial Sidband (VSB) Modulation. Noise in Linear Receivers, Noise in AM Receivers. Frequency-Division Multiplexing. |
| | Chapters 4 and 9. Angle Modulation<br>Frequency Modulation (FM), Phase Modulation (PM).Generation of FM wave. Demodulation of FM wave. Noise in FM Receivers. |
| | MIDTERM 2 (75 MINUTES) (Test on modulation.) |
| 1.5 | Chapter 5. Pulse Modulation: Transition from Analog to Digital Communication.<br>Sampling Process. Pulse-Amplitude Modulation (PAM). Quantization Process. Pulse-Code Modulation (PCM). Time-Division Multiplexing, Digital Multiplexers. Delta Modulation. |

| | |
|---|---|
| 1.0 | Chapter 6. Baseband Tranmission<br>Intersymbol Interference, Nyquist's Criterion for Distortionless Transmission, Baseband M-ary PAM Transmission, Optimum Linear Receiver. |
| 1.0 | Chapter 7. Passband Digital Transmission<br>Coherent Phase-Shift Keying, Coherent Frequency-Shift Keying, Hybrid Amplitude/Phase Modulation, Detection of Signals with Unknown Phase. Noncoherent Orthogonal Modulation, Differential Phase-Shift Keying. |
| 0.5 | FINAL EXAM (110 MINUTES) |

**Grading Policy**

(1) Grades will be assigned based on the class curve.

(2) A performance around the average class performance will earn a B-; a performance superior to the class mean will earn a B or B+ and a very superior performance will gain an A- or A. A performance inferior to the class mean will earn a C and a very inferior performance a D or an F.

| | |
|---|---|
| HOMEWORK (including computer work) | 12% |
| MIDTERMS | 53% |
| FINAL EXAM | 35% |

EXAMS CANNOT BE MISSED.

HOMEWORK WILL BE ASSIGNED EVERY THURSDAY AND WILL BE DUE THE FOLLOWING THURSDAY.

HOMEWORK MUST BE TURNED IN ON TIME AND CLEAN FORMAT.

COURSE LEARNING OBJECTIVES:

The course is devoted to the study of principles of communication theory as applied to the transmission of information. The focus is on the basic issues, relating theory to practice wherever possible. At the end of this introductory course in communication, student should understand and be able to apply the following to calculate and solve engineering problems in communication area:

1) Classical method for frequency analysis: Fourier transform and Fourier Series.

2) Spectral density and correlation functions of energy signals and power signals.

3) Using various techniques to find the energy and the power of a given signal.

4) Transmission of signals through linear filters and channel.

5) Hilbert transform and its application. Concept of pre-envelope, complex envelope and envelope and their applications.

6) Evaluating the response of a band-pass filter or channel to a band-pass signal.
7) Random processes. Transmission of a random process through a linear time invariant system. Gaussian process. Quadrature representation of a narrow-band noise.

8) Mathematical descriptions and the spectral characteristics of: amplitude modulation, frequency modulation and phase modulation. Frequency division multiplexing. Demodulation of AM, FM and PM signals.

9. Effect of noise in communication systems. Noise in CW modulation system. Noise in AM and FM receivers.

10. Sampling Theorem. Pulse-Amplitude Modulation (PAM). Pulse-Code Modulation (PCM). Quantization Process. Time Division Multiplexing (TDM).

11. Baseband Data Transmission. Band-pass data transmission. Digital modulation techniques such as PSK, FSK and ASK.

ASSESSENT OF STUDENTS' LEARNING:

At the end of the semester, the effect of this course on students' learning will be assessed based on the following criteria:

- The ability to apply knowledge of mathematics, science and engineering.
- The ability to design a system, component, or a process to meet desired needs.
- The ability to identify, formulate and solve engineering problems.
- A recognition of the need for, and an ability to engage in life-long learning.
- The ability to use the techniques, skills, and modern engineering tools necessary for engineering practice.

## 1.2 Text Book

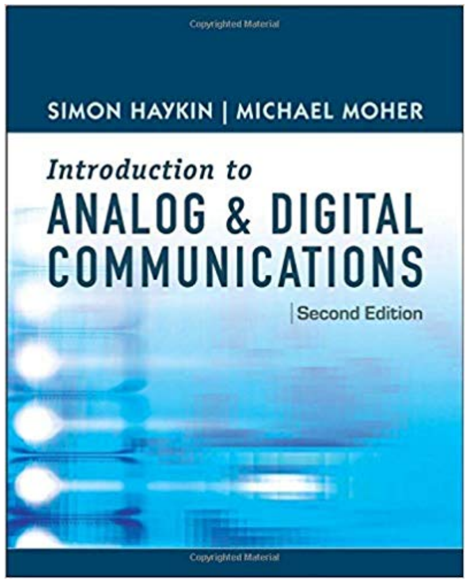



Figure 1.1: Official text book

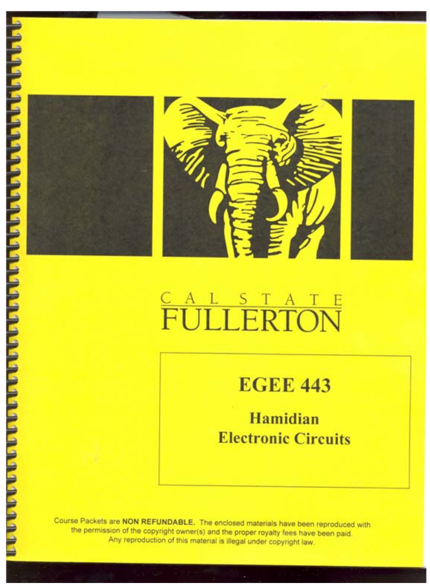



Figure 1.2: Instructor own text which we used more

## 1.3 Instructor contact information

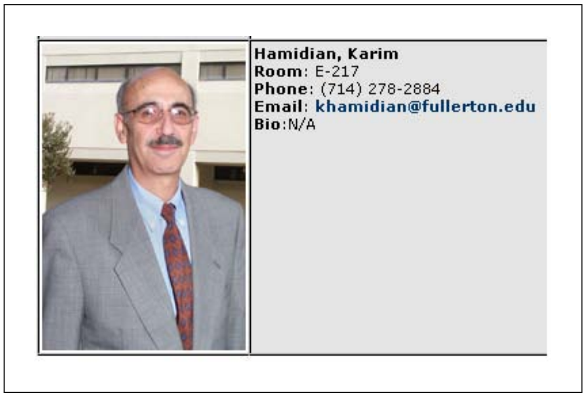
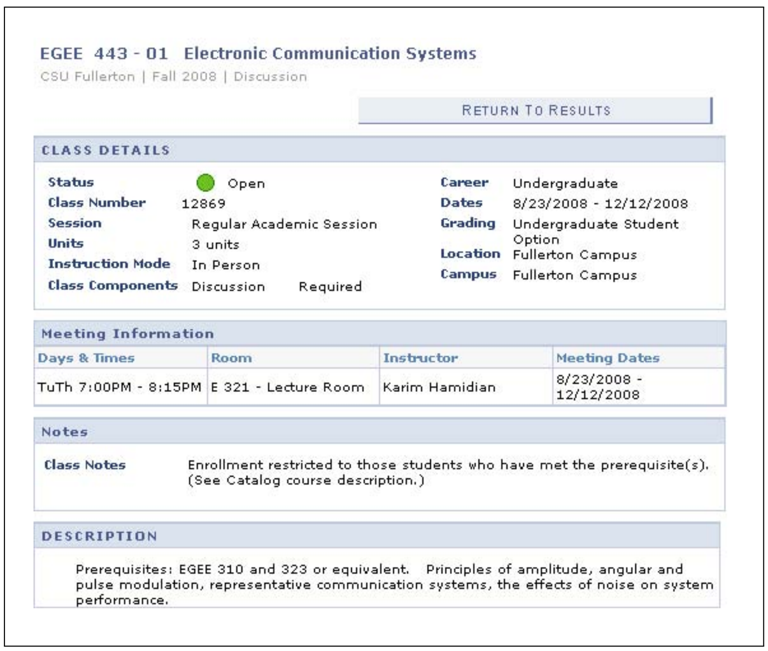



Figure 1.3: Professor Hamidian, Karim

## 1.4 Class information



EGEE 443 - 01 Electronic Communication Systems
CSU Fullerton | Fall 2008 | Discussion

RETURN TO RESULTS

CLASS DETAILS

| | | | |
|---|---|---|---|
| Status | Open | Career | Undergraduate |
| Class Number | 12869 | Dates | 8/23/2008 - 12/12/2008 |
| Session | Regular Academic Session | Grading | Undergraduate Student Option |
| Units | 3 units | Location | Fullerton Campus |
| Instruction Mode | In Person | Campus | Fullerton Campus |
| Class Components | Discussion Required | | |

Meeting Information

| Days & Times | Room | Instructor | Meeting Dates |
|---|---|---|---|
| TuTh 7:00PM - 8:15PM | E 321 - Lecture Room | Karim Hamidian | 8/23/2008 - 12/12/2008 |

Notes

Class Notes: Enrollment restricted to those students who have met the prerequisite(s). (See Catalog course description.)

DESCRIPTION

Prerequisites: EGEE 310 and 323 or equivalent. Principles of amplitude, angular and pulse modulation, representative communication systems, the effects of noise on system performance.

Figure 1.4: Course meeting time

# Chapter 2

# Handouts

## Local contents

## 2.1 Handout on random processes

EE 443 Chapter 1 Summary of chapter 1. 1

Random Signals

All useful message signals appear random to the receiver, since the receiver does not know, a priori, which of the possible message signals will be transmitted. Also, the noise superimposed to the desired signal is random. Therefore, we need an efficient description of random signals.

properties of a random variable, $X$:

1) The distribution function $F_X(x)$ of the random variable $X$ is given by:

$$F_X(x) \triangleq P_r[X \le x] \quad (33)$$

This is the probability that the value taken by the R.V. $X$ is less than or equal to a real number $x$.

$F_X(x)$ has the following properties:

a) $0 \le F_X(x) \le 1$

b) $F_X(x_1) \le F_X(x_2)$ if $x_1 \le x_2$

c) $F_X(-\infty) = 0$

d) $F_X(+\infty) = 1$

2) probability density function (Pdf) of the random variable $X$; $f_X(x)$:

EE 443. chapter 1 page 11.

The pdf and the distribution function of the R.V X are related to each other by:

$$f_X(x) = \frac{dF_X(x)}{dx} \qquad (34) \qquad \text{or}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(x_1)\,dx_1 \qquad (35)$$

Thus :

$$P[x_1 \le X \le x_2] = P[X \le x_2] - P[X \le x_1] = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(x)\,dx \qquad (36)$$

Properties of $f_X(x)$

a) $f_X(x) \geq 0$

b) $\int_{-\infty}^{+\infty} f_X(x)\,dx = 1$

3) Ensemble Averages :

a) The mean value $m_X$, of a continuous random variable X, is defined by:

$$m_X \triangleq E\{X\} = \int_{-\infty}^{+\infty} x f_X(x)\,dx \qquad (37)$$

b) The nth moment :

$$E\{X^n\} \triangleq \int_{-\infty}^{+\infty} x^n f_X(x)\,dx \qquad (38)$$

EE 443 : chapter 1 2

c) The mean square value or the power

$$P = E\{x^2\} \triangleq \int_{-\infty}^{+\infty} x^2 f_X(x)\,dx \quad (39)$$

d) Variance of the R.V. X ;

$$\sigma_X^2 = Var[X] \triangleq E\{(X - m_X)^2\} = \int_{-\infty}^{+\infty} (x - m_X)^2 f_X(x)\,dx \quad (40)$$

The variance $\sigma_X^2$ is a measure of the randomness of the random variable X.

e) One can verify that :

$E\{(X-\mu)^2\} = E\{X^2 + \mu^2 - 2X\mu\}$
$= E\{X^2\} + E\{\mu^2\} - 2\mu E\{X\}$
$= E\{X^2\} + \mu^2 - 2\mu^2$

$$\sigma_X^2 = E\{x^2\} - m_X^2 \quad (41)$$ where:

$E\{x^2\}$ represents the total power (average)
$m_X^2$ " the DC power and
$\sigma_X^2$ " " AC power (average)

## 4) Random Processes (R.P.)

A random process X(A,t) is a function of two variables ; a random event A and time. The following figure shows a R.P., which consist of N sample functions of time, $\{X_j(t)\}$. Each of the sample function can be viewed as the output of a different noise generator (That is distinct but identical noise generators).

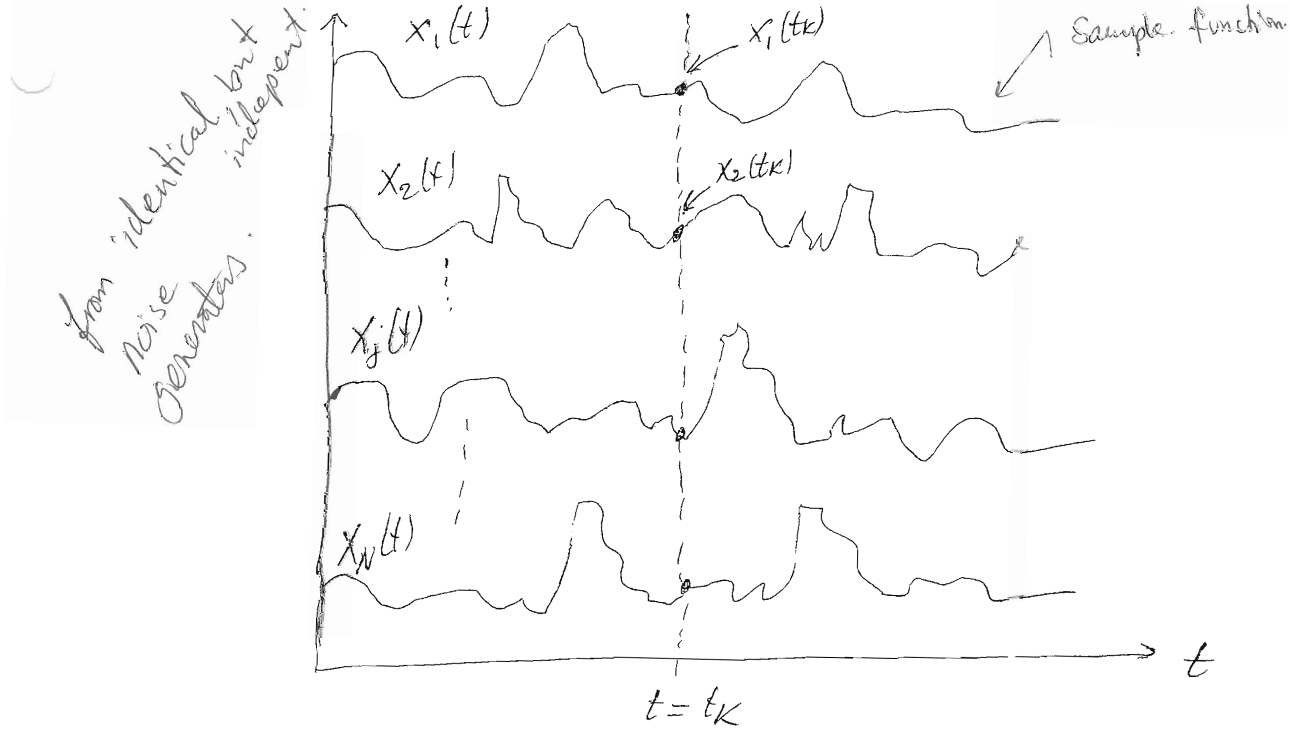
EE 443 chapter 1 3

- For a specific event $A_j$, that is when the event $A_j$ is known, we have a single function of time (sample function) $X(A_j, t) \triangleq X_j(t)$. The totality of all sample function is called an ensemble or R.P.

- For a specific time $t_K$, $X(A, t_K)$ is a random variab[le] $X(t_K)$ whose value depends on the event.

- For a specific event $A = A_j$ and a specific time $t = t_K$, $X(A_j, t_K)$ is a number.

From now on we will use $X(t)$ to describe the R.P. $X(A, t)$.




Random Noise Process

EE 443 chapter 1 - 4

next time

statistical Averages of a Random Process:

Because the value of a RP at any future time is unknown, a RP whose distribution function are continuous can be described statistically with a prob. density function (pdf). In general the form of the pdf of a RP will be different at different times. In most cases it is not possible to determine empirically the probability distribution function of a R.P. However, a partial description consisting of its mean and its autocorrelation function are sufficient for the needs of communication systems.

we define the mean of the R.P. $X(t)$ as:

$$m_X(t_k) = E\{X(t_k)\} \triangleq \int_{-\infty}^{+\infty} x f_{X_k}(x)\, dx \qquad (42)$$

where $X(t_k)$ is the Random variable obtained by observing the R.P. at time $t = t_k$ and $f_{X_k}(x)$ is the pdf over the ensemble of events at $t = t_k$.

b) Autocorrelation function of the R.P. $X(t)$:

$$R_X(t_1, t_2) \triangleq E\{X(t_1)X(t_2)\} \qquad (43)$$

where $X(t_1)$ and $X(t_2)$ are R. Variables obtained by observing the R.P. $X(t)$ at time $t_1$ and $t_2$.

$R_X(t_1, t_2)$ is a measure of similarity of the two samples $X(t_1)$ and $X(t_2)$ of the same R.P.

EE 443 chapter 1 5

stationarity :

(time Invariant ?)

a) A R.P. $x(t)$ is said to be stationary in strict sense (SSS) if none of its statistic are offected by a shift in the time origin.

b) A R.P. $x(t)$ is said to be stationary in wide sense (WSS) if its mean and its autocorrelation function do not change with a shift in the time origin. That is :

$$E\{x(t)\} = m_x = \text{constant}$$

$$R_x(t_1, t_2) = R_x(t_2 - t_1) \triangleq R_x(\tau)$$

Note that : $(SSS) \underset{\not\leftarrow}{\longrightarrow} WSS$

The autocorrelation of WSS is defined as :

$$R_x(\tau) \triangleq E\{x(t)x^*(t+\tau)\} \quad \text{or}$$

$$R_x(\tau) \triangleq E\{x(t)x^*(t-\tau)\} \qquad (44)$$

Properties of $R_x(\tau)$ for WSS :

1) $R_x(\tau) = R_x(-\tau)$ : If the R.P. $x(t)$ is real, then $R_x(\tau)$ is real and even.

2) $R_x(\tau) \leq R_x(0)$

3) $R_x(\tau) \overset{F.T}{\longleftrightarrow} S_x(f)$ P.S.D

4) $R_x(0) = E\{x^2(t)\} \triangleq P_{av}$

EE 443 chapter 1 6

## Time Averaging and Ergodicity:

To find $m_X$ and $R_X(\tau)$ by ensemble averaging, we have to average over all Sample function of the R.P. This would require the Knowledge of I° and II° order joint probability density functions, which are not generally available.

We will consider a particular class of R.P. Known as ergodic process, where its time averages equal its ensemble averages, and its statistical properties can be obtained by time averaging a single Sample function of the process. Note for a R.P. to be ergodic it must be SSS.

a) we say a R.P. $X(t)$ is ergodic in mean if and only if:

$$m_X = E\{X(t)\} = \langle X(t) \rangle \quad (45) \quad \text{where}$$

$$E\{X(t)\} \triangleq \int_{-\infty}^{+\infty} x f_X(x)\,dx \quad (46) \quad \text{ensemble average}$$

$$\langle X(t) \rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} X(t)\,dt \quad (47) \quad \text{time average}$$

b) we say a R.P. $X(t)$ is ergodic in autocorrelation iff:

$$E\{X(t)X(t+\tau)\} = \langle X(t)X(t+\tau)\rangle \quad (48)$$

where:

$$R_X(\tau) = E\{X(t)X(t+\tau)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\, f_{X(t)X(t+\tau)}(x,y)\,dx\,dy \quad (49)$$

$$R_X(\tau) = \langle X(t)X(t+\tau)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} X(t)X(t+\tau)\,dt$$

EE 443 Chapter 1 7

For an ergodic process, fundamental electrical engineering parameters, such as dc value, rms value, and average power can be related to the moments of an ergodic process:

a) $m_X = E\{X(t)\}$ is the dc level of the signal

b) $m_X^2$ is the normalized power in the dc component

c) $E\{X^2(t)\}$ is the total average normalized powe

d) $\sqrt{E\{X^2(t)\}}$ is the root mean square (rms) value

e) $\sigma_X^2$ is the variance or the average normalized power in the ac component. where $\sigma_X^2 = E\{X^2(t)\} - m_X^2$.

Properties of the P.S.D of a R.P. X(t)

If the process X(t) is real then:

a) $S_X(f) \geq 0$ and is always real valued.

b) $S_X(f) = S_X(-f)$

c) $S_X(f) \overset{FT}{\longleftrightarrow} R_X(\tau)$

d) $P_X = \int_{-\infty}^{+\infty} S(f)\,df = R_X(0)$

Example; Consider the sample function $X(t) = A\cos(\omega t + \theta)$ where A and $f_0$ are constant and $\theta$ is a random variable uniformly distributed over $(0, 2\pi)$. That is:

EE 443 chapter 1 (8)

$$f_\theta(\alpha) = \begin{cases} \frac{1}{2\pi} & 0 \leq \alpha \leq 2\pi \\ 0 & \text{otherwise} \end{cases}$$

a) verify that $x(t)$ is ergodic in mean.

b) " " " " " " autocorrelation.

a) $$m_x = E\{x(t)\} = \int_{-\infty}^{+\infty} A\cos(\omega_0 t+\alpha) f_\theta(\alpha)d\alpha$$

$$= \frac{A}{2\pi}\int_{-\infty}^{+\infty}\cos(\omega_0 t+\alpha)d\alpha = 0$$

$$\langle x(t)\rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} A\cos(\omega_0 t+\theta)dt$$

$$= \lim_{T\to\infty} \frac{A}{T}\cdot\frac{1}{2\pi f_0}\left[\sin(2\pi f_0 t+\theta)\right]_{-T/2}^{T/2}$$

$$= \lim_{T\to\infty} \frac{A}{T}\cdot\frac{1}{2\pi f_0}\left\{\underbrace{\sin(\pi f_0 T+\theta) - \sin(-\pi f_0 T+\theta)}_{\leq 2}\right] = 0$$

Thus $E\{x(t)\} = \langle x(t)\rangle \Rightarrow$ Ergodic in mean.

b)

$$R_x(\tau) = E\{x(t)x(t+\tau)\}$$

$$= E\{A^2\cos(2\pi f_0 t+\theta)\cos(2\pi f_0 t + 2\pi f_0 \tau + \theta)\}$$



## 2.2 Cheat sheet

**Table A11.1** Summary of Properties of the Fourier Transform

| Property | Mathematical Description |
|---|---|
| 1. Linearity | $ag_1(t) + bg_2(t) \rightleftharpoons aG_1(f) + bG_2(f)$ where $a$ and $b$ are constants |
| 2. Time scaling | $g(at) \rightleftharpoons \frac{1}{\lvert a \rvert} G\left(\frac{f}{a}\right)$ where $a$ is a constant |
| 3. Duality | If $g(t) \rightleftharpoons G(f)$, then $G(t) \rightleftharpoons g(-f)$ |
| 4. Time shifting | $g(t - t_0) \rightleftharpoons G(f)\exp(-j2\pi f t_0)$ |
| 5. Frequency shifting | $\exp(j2\pi f_c t) g(t) \rightleftharpoons G(f - f_c)$ |
| 6. Area under $g(t)$ | $\int_{-\infty}^{\infty} g(t)\, dt = G(0)$ |
| 7. Area under $G(f)$ | $g(0) = \int_{-\infty}^{\infty} G(f)\, df$ |
| 8. Differentiation in the time domain | $\frac{d}{dt} g(t) \rightleftharpoons j2\pi f G(f)$ |
| 9. Integration in the time domain | $\int_{-\infty}^{t} g(\tau)\, d\tau \rightleftharpoons \frac{1}{j2\pi f} G(f) + \frac{G(0)}{2} \delta(f)$ |
| 10. Conjugate functions | If $g(t) \rightleftharpoons G(f)$, then $g^*(t) \rightleftharpoons G^*(-f)$ |
| 11. Multiplication in the time domain | $g_1(t) g_2(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(\lambda) G_2(f - \lambda)\, d\lambda$ |
| 12. Convolution in the time domain | $\int_{-\infty}^{\infty} g_1(\tau) g_2(t - \tau)\, d\tau \rightleftharpoons G_1(f) G_2(f)$ |

**Table A11.4** Trigonometric Identities

$\exp(\pm j\theta) = \cos\theta \pm j \sin\theta$

$\cos\theta = \frac{1}{2}[\exp(j\theta) + \exp(-j\theta)]$

$\sin\theta = \frac{1}{2j}[\exp(j\theta) - \exp(-j\theta)]$

$\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta - \sin^2\theta = \cos(2\theta)$

$\cos^2\theta = \frac{1}{2}[1 + \cos(2\theta)]$

$\sin^2\theta = \frac{1}{2}[1 - \cos(2\theta)]$

$2 \sin\theta \cos\theta = \sin(2\theta)$

$\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta$

$\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$

$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha \tan\beta}$

$\sin\alpha \sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$

$\cos\alpha \cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$

$\sin\alpha \cos\beta = \frac{1}{2}[\sin(\alpha - \beta) + \sin(\alpha + \beta)]$

# Chapter 3

# HWs

## Local contents

## 3.1 HW 1

### Local contents

### 3.1.1 Questions

EE 443 HW #1 (Chap. 2) page

due Thursday.

96 **Representation Of Signals And Systems**

**Problem** 2.1 ⟹ 2.19 in new Book.

(a) Find the Fourier transform of the half-cosine pulse shown in Fig. P2.4(*a*).
(b) Apply the time-shifting property to the result obtained in part (a) to evaluate the spectrum of the half-sine pulse shown in Fig. P2.4(*b*).
(c) What is the spectrum of a half-sine pulse having a duration equal to $aT$?
(d) What is the spectrum of the negative half-sine pulse shown in Fig. P2.4(*c*)?
(e) Find the spectrum of the single sine pulse shown in Fig. P2.4(*d*).

Hint: $g(t) = A\cos\left(\frac{\pi t}{T}\right) \cdot \text{rect}\left(\frac{t}{T}\right)$

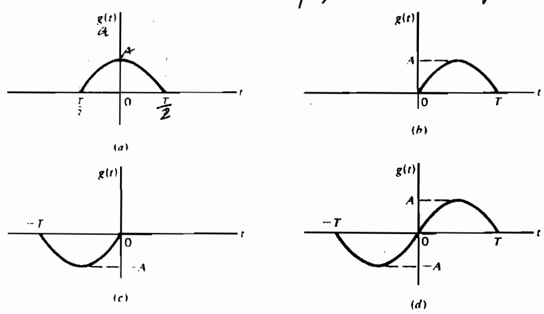



**Figure P2.4**

Prob. # 2.2

Given $g(t) = \exp(-t)\sin(2\pi f_c t)\,u(t)$. Find the Fourier Transform of $g(t)$: $F.T[g(t)] = ?$

2.3 ⟹ 2.20 in new Book.

**Problem** 2.3 Any function $g(t)$ can be split unambiguously into an *even part* and an *odd part*, as shown by

$$g(t) = g_e(t) + g_o(t) \quad \Rightarrow \quad g(t) = g_e(t) + g_o(t)$$

The even part is defined by

$$g_e(t) = \tfrac{1}{2}[g(t) + g(-t)]$$

and the odd part is defined by

$$g_o(t) = \tfrac{1}{2}[g(t) - g(-t)]$$

(a) Evaluate the even and odd parts of a rectangular pulse defined by

$$g(t) = A\,\text{rect}\left(\frac{t}{T} - \frac{1}{2}\right)$$

(That is find F.T. of $g_e(t)$ or $g_o(t)$)

(b) What are the Fourier transforms of these two parts of the pulse?

2.4

**Problem** Determine the inverse Fourier transform of the frequency function $G(f)$ defined by the amplitude and phase spectra shown in Fig. P . .

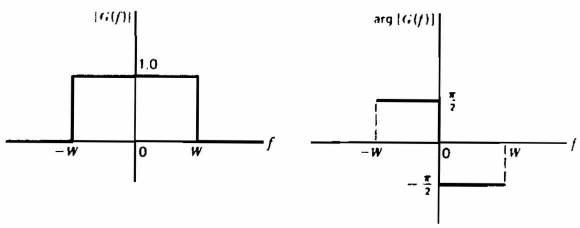



Figure P2.5

## 3.1.2 Problem 2.1

### 3.1.2.1 part(a)

Let $\mathcal{F}(g(t))$ be the Fourier Transform of $g(t)$, i.e. $\mathcal{F}(g(t)) = G(f)$. First we use the given hint and note that $g(t)$ can be written as follows

$$g(t) = A\cos\left(\frac{\pi t}{T}\right)\ rect\left(\frac{t}{T}\right)$$

Start by writing $\frac{\pi t}{T}$as $2\pi f_0 t$, where $f_0 = \frac{1}{2T}$. Now using the property that multiplication in time domain is the same as convolution in frequency domain, we obtain

$$G(f) = \mathcal{F}(A\cos(2\pi f_0 t)) \otimes \mathcal{F}\left(rect\left(\frac{t}{T}\right)\right) \tag{1}$$

But

$$\begin{aligned}\mathcal{F}(A\cos(2\pi f_0 t)) &= A\ \mathcal{F}(\cos(2\pi f_0 t))\\ &= A\ \mathcal{F}\left(\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}\right)\\ &= \frac{A}{2}\ \mathcal{F}\left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right)\\ &= \frac{A}{2}\left[\ \mathcal{F}\left(e^{j2\pi f_0 t}\right) +\ \mathcal{F}\left(e^{-j2\pi f_0 t}\right)\right]\end{aligned}$$

But $\mathcal{F}\left(e^{j2\pi f_0 t}\right) = \delta(f - f_0)$ and $\mathcal{F}\left(e^{-j2\pi f_0 t}\right) = \delta(f + f_0)$hence the above becomes

$$\mathcal{F}(A\cos(2\pi f_0 t)) = \frac{A}{2}[\ \delta(f - f_0) +\ \delta(f + f_0)] \tag{2}$$

Substitute (2) into (1) we obtain

$$G(f) = \frac{A}{2}[\ \delta(f - f_0) +\ \delta(f + f_0)] \otimes \mathcal{F}\left(rect\left(\frac{t}{T}\right)\right)$$

But $\mathcal{F}\left(rect\left(\frac{t}{T}\right)\right) = T\,\text{sinc}(fT)$, hence the above becomes

$$\mathcal{F}(g(t)) = \frac{A}{2}[\ \delta(f - f_0) +\ \delta(f + f_0)] \otimes T\,\text{sinc}(fT)$$

Now using the property of convolution with a delta, we obtain

$$\boxed{G(f) = \frac{AT}{2}[\ \text{sinc}((f - f_0)T) +\ \text{sinc}((f + f_0)T)]}$$

note: by doing more trigonometric manipulations, the above can be written as

$$\boxed{G(f) = \frac{2AT\cos(\pi fT)}{\pi(1 - 4f^2T^2)}}$$

#### 3.1.2.2 part(b)

Apply the time shifting property $g(t) \Longleftrightarrow G(f)$, hence $g(t-t_0) \Longleftrightarrow e^{-j2\pi f t_0} G(f)$

From part(a) we found that $F(g(t)) = \frac{AT}{2}[\operatorname{sinc}((f-f_0)T) + \operatorname{sinc}((f+f_0)T)]$, so in this part, the function in part(a) is shifted in time to the right by amount $\frac{T}{2}$, let the new function be $h(t)$ ,hence we need to multiply $G(f)$ by $e^{-j2\pi f\frac{T}{2}}$ ,hence

$$\begin{aligned}
F\left(g\left(t-\frac{T}{2}\right)\right) &= F(h(t)) \\
&= H(f) \\
&= e^{-j\pi fT}\left(\frac{AT}{2}[\operatorname{sinc}((f-f_0)T) + \operatorname{sinc}((f+f_0)T)]\right)
\end{aligned}$$

#### 3.1.2.3 part(c)

Using the time scaling property $g(t) \Longleftrightarrow G(f)$, hence $g(at) \Longleftrightarrow \frac{1}{|a|}G\left(\frac{f}{a}\right)$, and since we found in part(b) that $H(f) = e^{-j\pi fT}\left(\frac{AT}{2}[\operatorname{sinc}((f-f_0)T) + \operatorname{sinc}((f+f_0)T)]\right)$, hence

$$\boxed{F\{h(at)\} = \frac{1}{|a|}e^{-j\pi\frac{f}{a}T}\left(\frac{AT}{2}\left[\operatorname{sinc}\left(\left(\frac{f}{a}-f_0\right)T\right) + \operatorname{sinc}\left(\left(\frac{f}{a}+f_0\right)T\right)\right]\right)}$$

#### 3.1.2.4 part(d)

Let $f(t)$ be the function which is shown in figure 2.4c, we see that

$$f(t) = -h(-t)$$

where $h(t)$ is the function shown in figure 2.4(b). We found in part(b) that

$$H(f) = e^{-j\pi fT}\left(\frac{AT}{2}[\operatorname{sinc}((f-f_0)T) + \operatorname{sinc}((f+f_0)T)]\right)$$

Now using the property that $h(t) \Longleftrightarrow H(f)$ then $h(-t) \Longleftrightarrow \frac{1}{|-1|}H(-f) = H(-f)$, hence

$$\boxed{F\{f(t)\} = -e^{j\pi fT}\left(\frac{AT}{2}[\operatorname{sinc}((-f-f_0)T) + \operatorname{sinc}((-f+f_0)T)]\right)}$$

#### 3.1.2.5 part(e)

This function, call it $g_1(t)$ , is the sum of the functions shown in figure 2.4(b) and figure 2.4(c), then the Fourier transform of $g_1(t)$ is the sum of the Fourier transforms of the functions in these two figures (using the linearity of the Fourier transforms). Hence

$$\begin{aligned}
F(g_1(t)) = e^{-j\pi fT}&\left(\frac{AT}{2}[\operatorname{sinc}((f-f_0)T) + \operatorname{sinc}((f+f_0)T)]\right) \\
&- e^{j\pi fT}\left(\frac{AT}{2}[\operatorname{sinc}((-f-f_0)T) + \operatorname{sinc}((-f+f_0)T)]\right)
\end{aligned}$$

The above can be simplified to

$$\begin{aligned}\mathcal{F}(g_1(t)) &= \frac{AT}{2}\left(\operatorname{sinc}((f+f_0)T)\left[e^{j\pi fT} + e^{-j\pi fT}\right] + \operatorname{sinc}((f-f_0)T)\left[e^{j\pi fT} + e^{-j\pi fT}\right]\right)\\ &= \frac{AT}{2}\left(\operatorname{sinc}((f+f_0)T)[2\cos(\pi fT)] + \operatorname{sinc}((f-f_0)T)[2\cos(\pi fT)]\right)\end{aligned}$$

Hence

$$\boxed{\mathcal{F}(g_1(t)) = AT\cos(\pi fT)[\operatorname{sinc}((f+f_0)T) + \operatorname{sinc}((f-f_0)T)]}$$

### 3.1.3 Problem 2.2

Given $g(t) = e^{-t}\sin(2\pi f_c t)u(t)$ find $\mathcal{F}(g(t))$ Answer:

$$\mathcal{F}(g(t)) = \mathcal{F}\left(e^{-t}u(t)\right) \otimes \mathcal{F}(\sin(2\pi f_c t)) \tag{1}$$

But

$$\mathcal{F}(\sin(2\pi f_0 t)) = \frac{1}{2j}[\delta(f-f_c) - \delta(f+f_c)] \tag{2}$$

and

$$\begin{aligned}\mathcal{F}\left(e^{-t}u(t)\right) &= \int_0^\infty e^{-t}e^{-j2\pi ft}dt = \int_0^\infty e^{-t(1+j2\pi f)}dt\\ &= \frac{\left[e^{-t(1+j2\pi f)}\right]_0^\infty}{-(1+j2\pi f)} = \frac{0-1}{-(1+j2\pi f)}\\ &= \frac{1}{1+j2\pi f}\end{aligned} \tag{3}$$

Substitute (2) and (3) into (1) we obtain

$$\begin{aligned}\mathcal{F}(g(t)) &= \frac{1}{2j}[\delta(f-f_c) - \delta(f+f_c)] \otimes \frac{1}{1+j2\pi f}\\ &= \frac{1}{2j}\left[\frac{1}{1+j2\pi(f-f_c)} - \frac{1}{1+j2\pi(f+f_c)}\right]\end{aligned}$$

### 3.1.4 Problem 2.3

#### 3.1.4.1 part(a)

$$\begin{aligned}g(t) &= A\ rect\left(\frac{t}{T} - \frac{1}{2}\right)\\ &= A\ rect\left(\frac{t-\frac{T}{2}}{T}\right)\end{aligned}$$

hence it is a rect function with duration $T$ and centered at $\frac{T}{2}$ and it has height $A$

$$g_e = \frac{g(t) + g(-t)}{2} \tag{1}$$
$$g_o = \frac{g(t) - g(-t)}{2}$$

Hence $g_e = \frac{1}{2}\left[A\ rect\left(\frac{t}{T} - \frac{1}{2}\right) + A\ rect\left(\frac{-t}{T} - \frac{1}{2}\right)\right]$ which is a rectangular pulse of duration $2T$ and centered at zero and height $A$

$g_o = \frac{1}{2}\left[A\ rect\left(\frac{t}{T} - \frac{1}{2}\right) - A\ rect\left(\frac{-t}{T} - \frac{1}{2}\right)\right]$ which is shown in the figure below

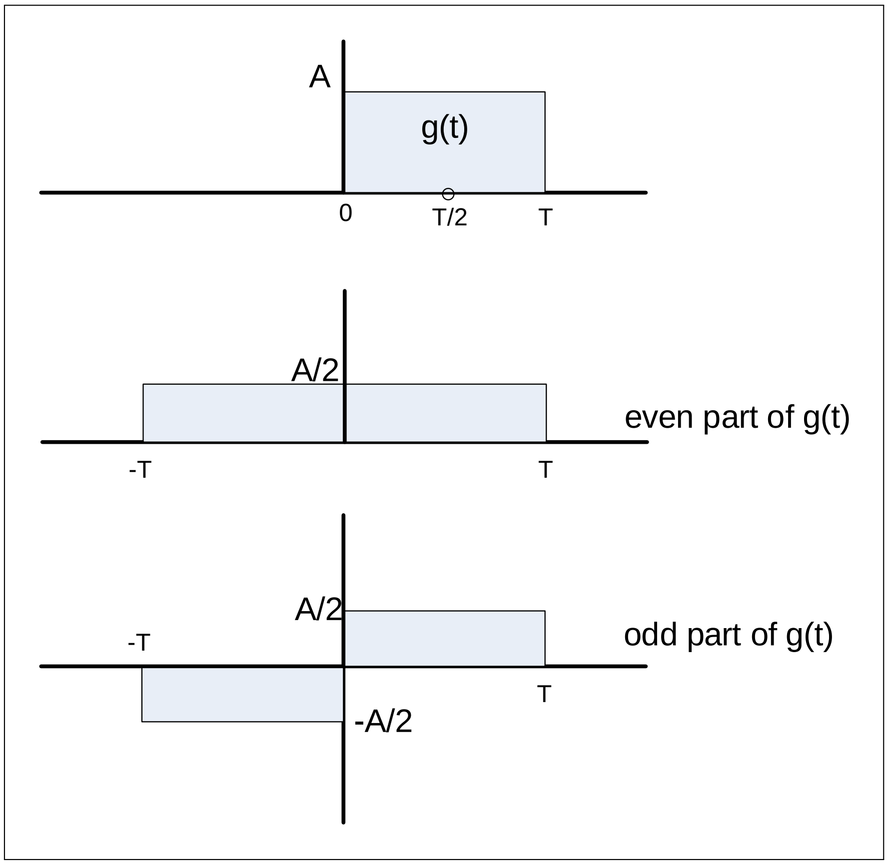



Figure 3.1: rectangular pulse

#### 3.1.4.2 part(b)

$$\begin{aligned} F\left(g(t)\right) &= F\left(A\ rect\left(\frac{t-\frac{T}{2}}{T}\right)\right) \\ &= AT\ \operatorname{sinc}(fT)\ e^{-j2\pi f\frac{T}{2}} \\ &= AT\ \operatorname{sinc}(fT)\ e^{-j\pi fT} \end{aligned} \tag{2}$$

Now using the property that $g(t) \Leftrightarrow G(f)$, then $g(-t) \Leftrightarrow G(-f)$, then we write

$$\begin{aligned} F(g(-t)) &= G(-f) \\ &= AT\ \operatorname{sinc}(-fT)\ e^{j\pi fT} \end{aligned} \tag{3}$$

Now, using linearity of Fourier transform, then from (1) we obtain

$$\begin{aligned} F(g_e(t)) &= F\left(\frac{g(t)+g(-t)}{2}\right) \\ &= \frac{1}{2}\left[F(g(t)) + F(g(-t))\right] \\ &= \frac{1}{2}\left[AT\ \operatorname{sinc}(fT)\ e^{-j\pi fT} + AT\ \operatorname{sinc}(-fT)\ e^{j\pi fT}\right] \\ &= \frac{AT}{2}\left[\operatorname{sinc}(fT)\ e^{-j\pi fT} + \operatorname{sinc}(-fT)\ e^{j\pi fT}\right] \end{aligned}$$

now $\operatorname{sinc}(-fT) = \frac{\sin(-\pi fT)}{-\pi fT} = \frac{-\sin(\pi fT)}{-\pi fT} = \operatorname{sinc}(fT)$, hence the above becomes

$$\begin{aligned} F(g_e(t)) &= \frac{AT\operatorname{sinc}(fT)}{2}\left[\ e^{-j\pi fT} +\ e^{j\pi fT}\right] \\ &= \frac{AT\operatorname{sinc}(fT)}{2}\left[\ 2\cos(\pi fT)\right] \end{aligned}$$

Hence

$$\boxed{F(g_e(t)) = AT\operatorname{sinc}(fT)\cos(\pi fT)}$$

Now to find the Fourier transform of the odd part

$$g_o = \frac{g(t)-g(-t)}{2}$$

Hence

$$\begin{aligned} F(g_o(t)) &= F\left(\frac{g(t)-g(-t)}{2}\right) \\ &= \frac{1}{2}\left[F(g(t)) - F(g(-t))\right] \\ &= \frac{1}{2}\left[AT\ \operatorname{sinc}(fT)\ e^{-j\pi fT} - AT\ \operatorname{sinc}(-fT)\ e^{j\pi fT}\right] \\ &= \frac{AT}{2}\left[\operatorname{sinc}(fT)\ e^{-j\pi fT} - \operatorname{sinc}(fT)\ e^{j\pi fT}\right] \\ &= \frac{AT\operatorname{sinc}(fT)}{2}\left[\ e^{-j\pi fT} -\ e^{j\pi fT}\right] \\ &= \frac{-AT\operatorname{sinc}(fT)}{2}\left[\ e^{j\pi fT} - e^{-j\pi fT}\right] \\ &= \frac{-AT\operatorname{sinc}(fT)}{2}\left[\ 2j\sin(\pi fT)\right] \end{aligned}$$

Hence

$$\boxed{F\left(g_o\left(t\right)\right) = -jAT\operatorname{sinc}\left(fT\right)\sin\left(\pi fT\right)}$$

### 3.1.5 Problem 2.4

$$G\left(f\right) = \left|G\left(f\right)\right| e^{j\arg\left(G\left(f\right)\right)}$$

Hence from the diagram given, we write

$$G\left(f\right) = \begin{cases} 1\times e^{j\frac{\pi}{2}} & -W \leq f < 0 \\ 1\times e^{-j\frac{\pi}{2}} & 0 \leq f \leq W \end{cases}$$

Therefore, we can use a rect function now to express $G\left(f\right)$ over the whole $f$ range as follows

$$G\left(f\right) = e^{j\frac{\pi}{2}}\ rect\left(\frac{f+\frac{W}{2}}{W}\right) - e^{-j\frac{\pi}{2}} rect\left(\frac{f-\frac{W}{2}}{W}\right)$$

Now, noting that $\delta\left(t-t_0\right) \Leftrightarrow e^{-j2\pi t_0}$ and $\delta\left(t+t_0\right) \Leftrightarrow e^{j2\pi t_0}$ and $W\operatorname{sinc}\left(tW\right) \Leftrightarrow rect\left(\frac{f}{W}\right)$ and noting that shift in frequency by $\frac{W}{2}$becomes multiplication by $e^{-j2\pi t\frac{W}{2}}$, then now we write

$$\begin{aligned} g\left(t\right) &= F^{-1}\left(e^{j\frac{\pi}{2}}\ rect\left(\frac{f+\frac{W}{2}}{W}\right)\right) - F^{-1}\left(e^{-j\frac{\pi}{2}} rect\left(\frac{f-\frac{W}{2}}{W}\right)\right) \\ &= F^{-1}\left(e^{j\frac{\pi}{2}}\right)\ \otimes F^{-1}\left(rect\left(\frac{f+\frac{W}{2}}{W}\right)\right) - F^{-1}\left(e^{-j\frac{\pi}{2}}\right)\ \otimes F^{-1}\left(rect\left(\frac{f-\frac{W}{2}}{W}\right)\right) \end{aligned}$$

Hence

$$\begin{aligned} g\left(t\right) &= \left[\delta\left(t+\frac{\pi}{2}\right)\otimes W\operatorname{sinc}\left(tW\right)e^{-j2\pi t\frac{W}{2}}\right] - \left[\delta\left(t-\frac{\pi}{2}\right)\otimes W\operatorname{sinc}\left(tW\right)e^{j2\pi t\frac{W}{2}}\right] \\ &= W\operatorname{sinc}\left(\left(t+\frac{\pi}{2}\right)W\right)e^{-j2\pi\left(t+\frac{\pi}{2}\right)\frac{W}{2}} - W\operatorname{sinc}\left(\left(t-\frac{\pi}{2}\right)W\right)e^{j2\pi\left(t-\frac{\pi}{2}\right)\frac{W}{2}} \\ &= W\operatorname{sinc}\left(\left(t+\frac{\pi}{2}\right)W\right)e^{-j\pi Wt-j\pi W\frac{\pi}{2}} - W\operatorname{sinc}\left(\left(t-\frac{\pi}{2}\right)W\right)e^{j\pi Wt-j\pi W\frac{\pi}{2}} \end{aligned}$$

Hence

$$\boxed{g\left(t\right) = We^{-\frac{j\pi^2W}{2}}\left(\operatorname{sinc}\left(\left(t+\frac{\pi}{2}\right)W\right)e^{-j\pi Wt} - \operatorname{sinc}\left(\left(t-\frac{\pi}{2}\right)W\right)e^{j\pi Wt}\right)}$$

### 3.1.6 Key solution

EE 443 HW#1 Key

page 1

1

**CHAPTER 2**

**Representation of Signals and Systems**

Problem 2.1

(a) The half-cosine pulse g(t) of Fig. P2.1(a) may be considered as the product of the rectangular function rect(t/T) and the sinusoidal wave A cos(πt/T). Since

$$\text{rect}\left(\frac{t}{T}\right) \rightleftharpoons T\,\text{sinc}(fT)$$

$$A\cos\left(\frac{\pi t}{T}\right) \rightleftharpoons \frac{A}{2}\left[\delta\left(f-\frac{1}{2T}\right)+\delta\left(f+\frac{1}{2T}\right)\right]$$

and multiplication in the time domain is transformed into convolution in the frequency domain, it follows that

$$G(f) = [T\,\text{sinc}(fT)] \star \left\{\frac{A}{2}\left[\delta\left(f-\frac{1}{2T}\right)+\delta\left(f+\frac{1}{2T}\right)\right]\right\}$$

where $\star$ denotes convolution. Therefore, noting that

$$\text{sinc}(fT) \star \delta\left(f-\frac{1}{2T}\right) = \text{sinc}\left[T\left(f-\frac{1}{2T}\right)\right]$$

$$\text{sinc}(fT) \star \delta\left(f+\frac{1}{2T}\right) = \text{sinc}\left[T\left(f+\frac{1}{2T}\right)\right]$$

we obtain the desired result

$$G(f) = \frac{AT}{2}\left[\text{sinc}\left(fT-\frac{1}{2}\right)+\text{sinc}\left(fT+\frac{1}{2}\right)\right]$$

(b) The half-sine pulse of Fig. P2.1(b) may be obtained by shifting the half-cosine pulse to the right by T/2 seconds. Since a time shift of T/2 seconds is equivalent to multiplication by exp(-jπfT) in the frequency domain, it follows that the Fourier transform of the half-sine pulse is

$$G(f) = \frac{AT}{2}\left[\text{sinc}\left(fT-\frac{1}{2}\right)+\text{sinc}\left(fT+\frac{1}{2}\right)\right]\exp(-j\pi fT)$$

(c) The Fourier transform of a half-sine pulse of duration aT is equal to

$$\frac{|a|AT}{2}\left[\text{sinc}\left(afT-\frac{1}{2}\right)+\text{sinc}\left(afT+\frac{1}{2}\right)\right]\exp(-j\pi afT)$$

(d) The Fourier transform of the negative half-sine pulse shown in Fig. P2.1(c) is obtained from the result of part (c) by putting a= -1, and multiplying the result by -1, and so we find that its Fourier transform is equal to

EE 443 HW#1 page 2

2

$$= \frac{AT}{2}[\text{sinc}(fT+\frac{1}{2}) + \text{sinc}(fT-\frac{1}{2})]\exp(j\pi fT)$$

(e) The full-sine pulse of Fig. P2.1(d) may be considered as the superposition of the half-sine pulses shown in parts (b) and (c) of the figure. The Fourier transform of this pulse is therefore

$$G(f) = \frac{AT}{2}[\text{sinc}(fT-\frac{1}{2}) + \text{sinc}(fT+\frac{1}{2})][\exp(-j\pi fT)-\exp(j\pi fT)]$$

$$= -jAT[\text{sinc}(fT-\frac{1}{2}) + \text{sinc}(fT+\frac{1}{2})]\sin(\pi fT)$$

$$= -jAT\left[\frac{\sin(\pi fT-\frac{\pi}{2})}{\pi fT-\frac{\pi}{2}} + \frac{\sin(\pi fT+\frac{\pi}{2})}{\pi fT+\frac{\pi}{2}}\right]\sin(\pi fT)$$

$$= -jAT[-\frac{\cos(\pi fT)}{\pi fT-\frac{\pi}{2}} + \frac{\cos(\pi fT)}{\pi fT+\frac{\pi}{2}}]\sin(\pi fT)$$

$$= jAT[\frac{\sin(2\pi fT)}{2\pi fT-\pi} - \frac{\sin(2\pi fT)}{2\pi fT+\pi}]$$

$$= jAT[-\frac{\sin(2\pi fT-\pi)}{2\pi fT-\pi} + \frac{\sin(2\pi fT+\pi)}{2\pi fT+\pi}]$$

$$= jAT[\text{sinc}(2fT+1) - \text{sinc}(2fT-1)]$$

Problem 2.2

Consider next an exponentially damped sinusoidal wave defined by (see Fig. 1):

$$g(t) = \exp(-t)\sin(2\pi f_c t)u(t)$$

In this case, we note that

$$\sin(2\pi f_c t) = \frac{1}{2j}[\exp(j2\pi f_c t) - \exp(-j2\pi f_c t)]$$

Therefore, applying the frequency-shifting property to the Fourier transform pair we find that the Fourier transform of the damped sinusoidal wave of Fig. 1 is

$$G(f) = \frac{1}{2j}\left[\frac{1}{1+j2\pi(f-f_c)} - \frac{1}{1+j2\pi(f+f_c)}\right]$$

$$= \frac{2\pi f_c}{(1+j2\pi f)^2 + (2\pi f_c)^2}$$

EE 443 HW #1 page 3

3

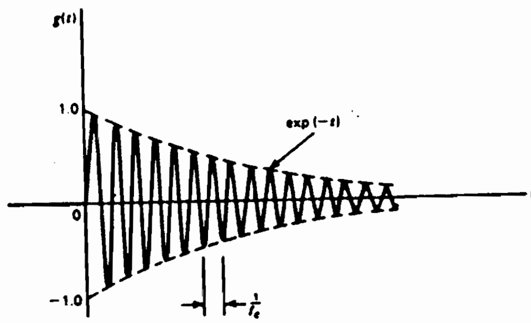



**Figure 1** Damped sinusoidal wave.

**Problem 2.3**

(a) The even part $g_e(t)$ of a pulse $g(t)$ is given by

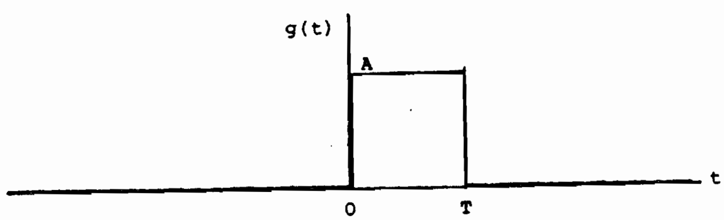
$$g_e(t) = \frac{1}{2}[g(t) + g(-t)]$$

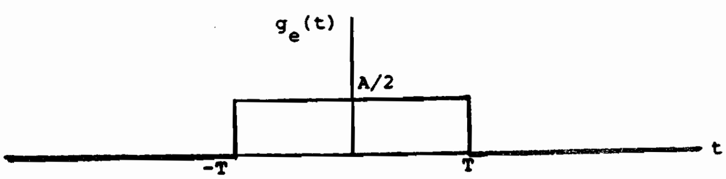
Therefore, for $g(t) = A\,\mathrm{rect}(\frac{t}{T} - \frac{1}{2})$, we obtain

$$g_e(t) = \frac{A}{2}[\mathrm{rect}(\frac{t}{T} - \frac{1}{2}) + \mathrm{rect}(-\frac{t}{T} - \frac{1}{2})]$$

$$= \frac{A}{2}[\mathrm{rect}(\frac{t}{2T})]$$

which is shown illustrated below:



EE 443 HW #1 page 4

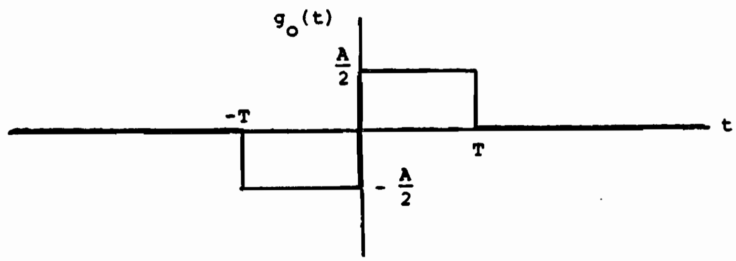
4

ıe odd part of g(t) is defined by

$$g_o(t) = \frac{1}{2}[g(t) - g(-t)]$$

$$= \frac{A}{2}[\mathrm{rect}(\frac{t}{T} - \frac{1}{2}) - \mathrm{rect}\ (-\frac{t}{T} - \frac{1}{2})]$$

which is illustrated below:




(b) The Fourier transform of the even part is

$$G_e(f) = AT\ \mathrm{sinc}(2fT)$$

The Fourier transform of the odd part is

$$G_o(f) = \frac{AT}{2}\mathrm{sinc}(fT)\exp(-j\pi fT)$$
$$- \frac{AT}{2}\mathrm{sinc}(fT)\exp(j\pi fT)$$
$$= \frac{AT}{j}\mathrm{sinc}(fT)\sin(\pi fT)$$

Problem 2.4

$$G(f) = \begin{cases} \exp(j\frac{\pi}{2}), & -W \le f \le 0 \\ \exp(-j\frac{\pi}{2}), & 0 \le f \le W \\ 0, & \text{otherwise} \end{cases}$$

Therefore, applying the formula for the inverse Fourier transform, we get

$$g(t) = \int_{-W}^{0} \exp(j\frac{\pi}{2})\exp(j2\pi ft)df + \int_{0}^{W} \exp(-j\frac{\pi}{2})\exp(j2\pi ft)dt$$

Replacing f with -f in the first integral and then interchanging the limits of integration:

EE 443 HW #1 page 5

5

$$g(t) = \int_0^W \exp(-j2\pi ft + j\frac{\pi}{2}) + \exp(j2\pi ft - j\frac{\pi}{2})]df$$

$$= 2\int_0^W \cos(2\pi ft - \frac{\pi}{2})df$$

$$= 2\int_0^W \sin(2\pi ft)df$$

$$= \left[-\frac{\cos(2\pi ft)}{\pi t}\right]_0^W$$

$$= \frac{1}{\pi t}[1-\cos(2\pi Wt)]$$

$$= \frac{2}{\pi t}\sin^2(\pi Wt)$$

## 3.2 HW 2

### Local contents

### 3.2.1 Questions

EE 443 Chapt. # 2 HW #2 page 1

**Problem 2.30** Determine and sketch the autocorrelation functions of the following exponential pulses:

(a) $g(t) = \exp(-at)u(t)$
✓ (b) $g(t) = \exp(-a|t|)$
✓ (c) $g(t) = \exp(-at)u(t) - \exp(at)u(-t)$

**Problem 2.32** Determine the autocorrelation function of the sinc pulse $A\,\text{sinc}(2Wt)$, and sketch it.

**Problem 2.33** The Fourier transform of a signal is defined by $|\text{sinc}(f)|$. Show that the autocorrelation function of this signal is triangular in form.

( Hint : Find $|G(f)|^2$, then find $R_g(\tau)$ )

**Problem 2.35** Consider a signal $g(t)$ defined by

$$g(t) = A_0 + A_1\cos(2\pi f_1 t + \theta) + A_2\cos(2\pi f_2 t + \theta)$$

(a) Determine the autocorrelation function $R_g(\tau)$ of this signal.
(b) What is the value of $R_g(0)$?
(c) Has any information about $g(t)$ been lost in obtaining the autocorrelation function?

( Hint Use Freq. domain approach. )

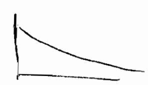
extra problem. do:

a) $\xi(t) \circledast \xi(t)$ where $\xi(t)$ is unit step function

b) $y(t) = t\,\xi(t) \circledast e^{at}\,\xi(t)$ $a<0$

c) $y(t) = u(t) \circledast h(t)$ where $u(t)$

and $h(t) = e^{-3t}u(t)$




### 3.2.2 Problem 2.30

Problem

Determine and sketch the autocorrelation function of the following

(b) $g(t) = e^{-a|t|}$

(c) $g(t) = e^{-at}u(t) - e^{at}u(-t)$

#### 3.2.2.1 part(b)

$$g(t) = \begin{cases} e^{-at} & t > 0 \\ 1 & t = 0 \\ e^{at} & t < 0 \end{cases}$$

Assume $a > 0$ for the integral to be defined. From definition, autocorrelation of a function $g(t)$ is

$$R(\tau) = \int_{-\infty}^{\infty} g(t)\, g^*(t-\tau)\, dt$$

Since $g(t)$ in this case is real, then $g^*(t-\tau) = g(t-\tau)$, hence

$$R(\tau) = \int_{-\infty}^{\infty} g(t)\, g(t-\tau)\, dt$$

Consider the 3 cases, $\tau < 0$ and $\tau > 0$ and when $\tau = 0$

case $\tau > 0$

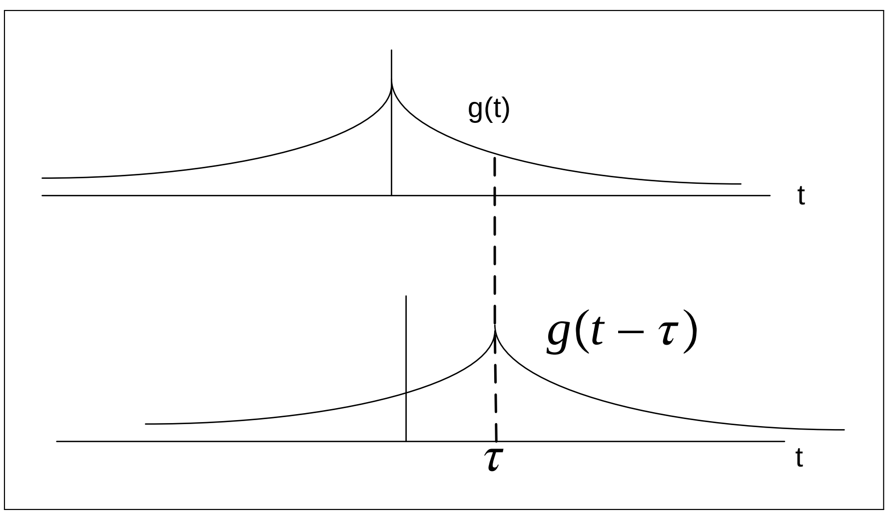



Figure 3.2: Case 1 Part b

Break the integral over the 3 regions, $\{-\infty, 0\}, \{0, \tau\}, \{\tau, \infty\}$

$$R(\tau) = \int_{-\infty}^{0} e^{at} e^{a(t-\tau)} dt + \int_{0}^{\tau} e^{-at} e^{a(t-\tau)} dt + \int_{\tau}^{\infty} e^{-at} e^{-a(t-\tau)} dt$$

But $\int_{-\infty}^{0} e^{at} e^{a(t-\tau)} dt = e^{-a\tau} \int_{-\infty}^{0} e^{2at} dt = e^{-a\tau} \frac{\left[e^{2at}\right]_{-\infty}^{0}}{2a} = e^{-a\tau} \frac{[1-0]}{2a} = \frac{e^{-a\tau}}{2a}$
and $\int_{0}^{\tau} e^{-at} e^{a(t-\tau)} dt = e^{-a\tau} \int_{0}^{\tau} 1 dt = \tau e^{-a\tau}$
and $\int_{\tau}^{\infty} e^{-at} e^{-a(t-\tau)} dt = e^{a\tau} \int_{\tau}^{\infty} e^{-2at} dt = e^{a\tau} \frac{\left[e^{-2at}\right]_{\tau}^{\infty}}{-2a} = e^{a\tau} \frac{\left[0-e^{-2a\tau}\right]}{-2a} = \frac{e^{-a\tau}}{2a}$
Hence for $\tau > 0$ we obtain

$$\begin{aligned} R(\tau) &= \frac{e^{-a\tau}}{2a} + \tau e^{-a\tau} + \frac{e^{-a\tau}}{2a} \\ &= \frac{e^{-a\tau}}{a} + \tau e^{-a\tau} \\ &= \boxed{e^{-a\tau}\left(\frac{1}{a} + \tau\right)} \end{aligned}$$

case $\tau < 0$

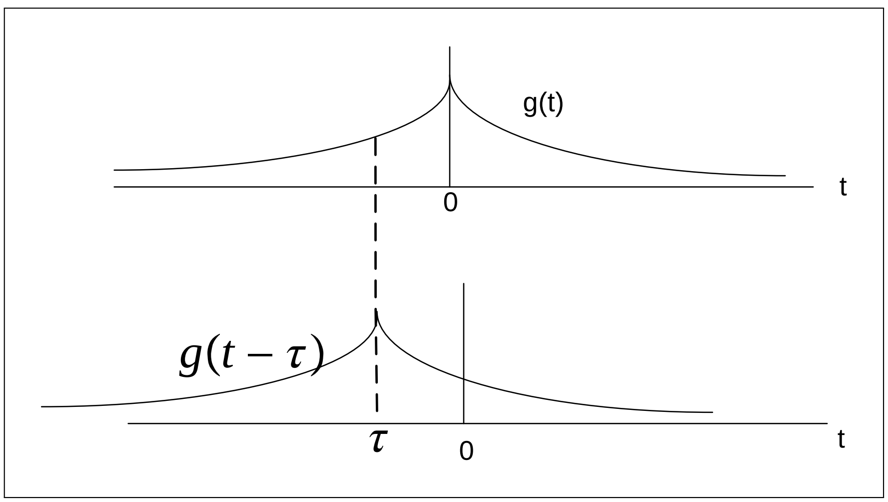



Figure 3.3: Case 2 Part b

Break the integral over the 3 regions, $\{-\infty, \tau\}, \{\tau, 0\}, \{0, \infty\}$

$$R(\tau) = \int_{-\infty}^{\tau} e^{at} e^{a(t-\tau)} dt + \int_{\tau}^{0} e^{-at} e^{a(t-\tau)} dt + \int_{0}^{\infty} e^{-at} e^{-a(t-\tau)} dt$$

Now $\int_{-\infty}^{\tau} e^{at} e^{a(t-\tau)} dt = e^{-a\tau} \int_{-\infty}^{\tau} e^{2at} dt = e^{-a\tau} \frac{\left[e^{2at}\right]_{-\infty}^{\tau}}{2a} = e^{-a\tau} \frac{\left[e^{2a\tau}-0\right]}{2a} = \frac{e^{a\tau}}{2a}$

and $\int_{\tau}^{0} e^{-at} e^{a(t-\tau)} dt = e^{-a\tau} \int_{\tau}^{0} 1 dt = -\tau e^{-a\tau}$

and $\int_{0}^{\infty} e^{-at} e^{-a(t-\tau)} dt = e^{a\tau} \frac{\left[e^{-2at}\right]_0^{\infty}}{-2a} = \frac{e^{a\tau}}{-2a}(0-1) = \frac{e^{a\tau}}{2a}$

Hence

$$\begin{aligned} R(\tau) &= \frac{e^{a\tau}}{2a} - \tau e^{-a\tau} + \frac{e^{a\tau}}{2a} \\ &= \boxed{e^{a\tau}\left(\frac{1}{a} - \tau\right)} \end{aligned}$$

When $\tau = 0$

$R(0)$ gives the the maximum power in the signal $g(t)$. Now evaluate this

$$\begin{aligned} R(\tau) &= \int_{-\infty}^{0} e^{at} e^{at} dt + \int_{0}^{\infty} e^{-at} e^{-at} dt \\ &= \frac{\left[e^{2at}\right]_{-\infty}^{0}}{2a} + \frac{\left[e^{-2at}\right]_{0}^{\infty}}{-2a} \\ &= \frac{1}{a} \end{aligned}$$

Hence

$$R(\tau) = \begin{cases} e^{-a\tau}\left(\frac{1}{a} + \tau\right) & \tau > 0 \\ \frac{1}{a} & \tau = 0 \\ e^{a\tau}\left(\frac{1}{a} - \tau\right) & \tau < 0 \end{cases}$$

Or we could write

$$\boxed{R(\tau) = e^{-|\tau| a}\left(\frac{1}{a} - (-|\tau|\right)}$$

This is a plot of $R(\tau)$, first plot is for $a = 1$ and the second for $a = 4$

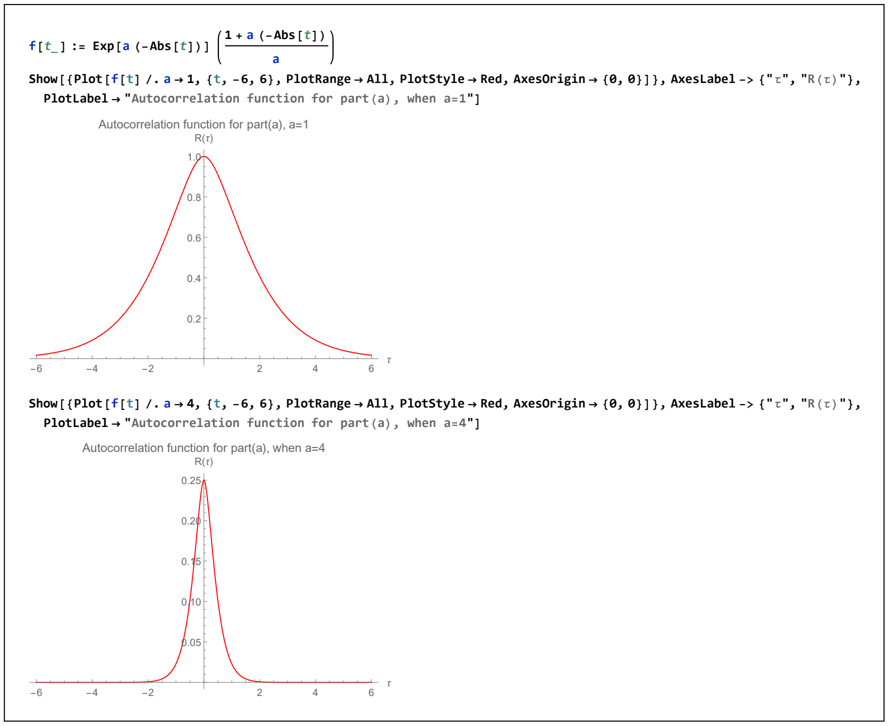



Figure 3.4: final part

#### 3.2.2.2 part(c)

$$g(t) = e^{-at}u(t) - e^{at}u(-t)$$

Assume $a > 0$.

Consider the 3 cases, $\tau < 0$ and $\tau > 0$ and when $\tau = 0$

$\boxed{\text{case } \tau > 0}$

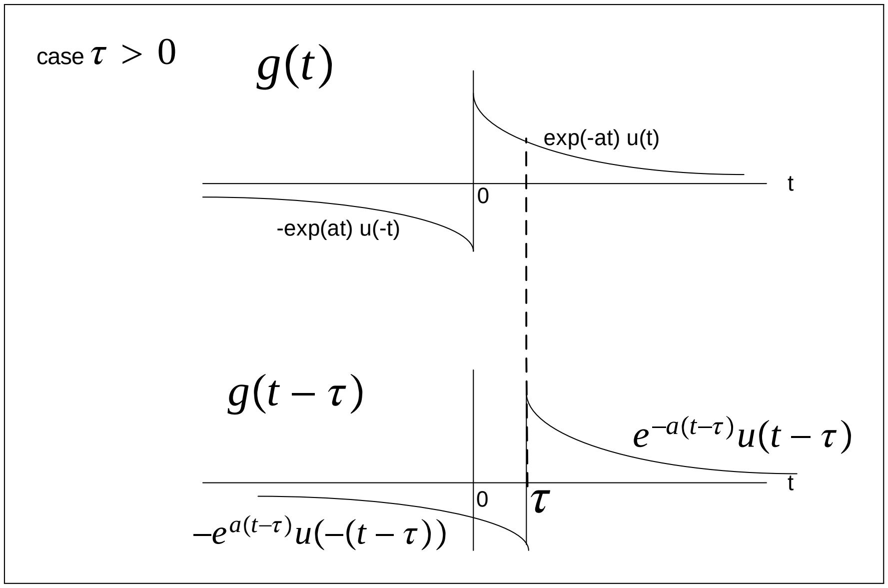



Figure 3.5: Case 1 Part c

Break the integral into 3 parts, $\{-\infty,0\},\{0,\tau\},\{\tau,\infty\}$

$$
\begin{aligned}
R(\tau) &= \int_{-\infty}^{0} g(t)\,g(t-\tau)\,dt + \int_{0}^{\tau} g(t)\,g(t-\tau)\,dt + \int_{\tau}^{\infty} g(t)\,g(t-\tau)\,dt \\
&= \int_{-\infty}^{0} -e^{at}\left(-e^{a(t-\tau)}\right)dt + \int_{0}^{\tau} e^{-at}\left(-e^{a(t-\tau)}\right)dt + \int_{\tau}^{\infty} e^{-at}\left(e^{-a(t-\tau)}\right)dt \\
&= e^{-a\tau}\int_{-\infty}^{0} e^{2at}dt - e^{-a\tau}\int_{0}^{\tau} 1dt + e^{a\tau}\int_{\tau}^{\infty} e^{-2at}dt \\
&= e^{-a\tau}\frac{\left[e^{2at}\right]_{-\infty}^{0}}{2a} - \tau e^{-a\tau} + e^{a\tau}\frac{\left[e^{-2at}\right]_{\tau}^{\infty}}{-2a} \\
&= e^{-a\tau}\frac{[1-0]}{2a} - \tau e^{-a\tau} + e^{a\tau}\frac{\left[0-e^{-2a\tau}\right]}{-2a} \\
&= \frac{e^{-a\tau}}{2a} - \tau e^{-a\tau} + \frac{e^{-a\tau}}{2a} \\
&= e^{-a\tau}\left(\frac{1}{2a} - \tau + \frac{1}{2a}\right) \\
&= e^{-a\tau}\left(\frac{1}{a} - \tau\right)
\end{aligned}
$$

case $\tau < 0$

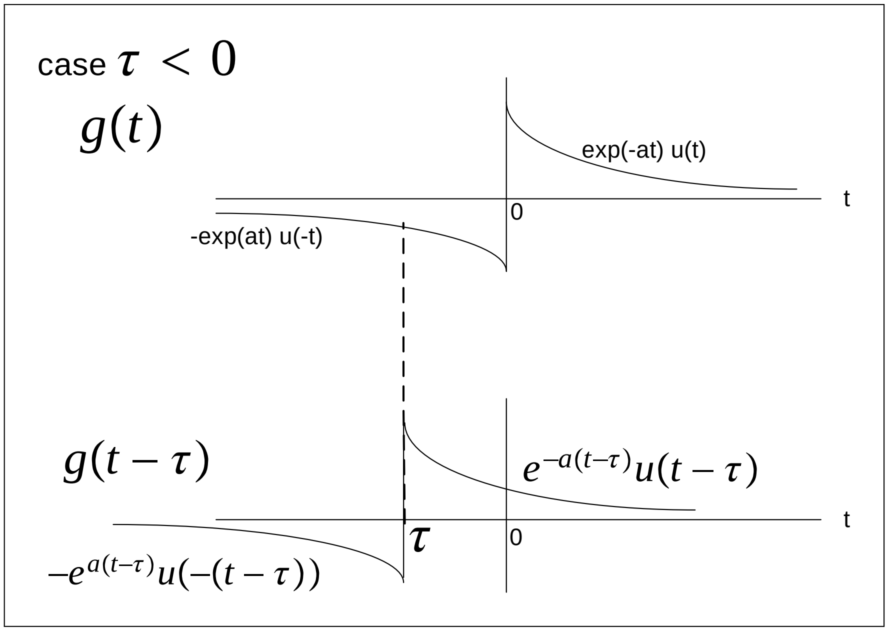



Figure 3.6: Case 2 Part c

Break the integral into 3 parts, $\{-\infty,\tau\},\{\tau,0\},\{0,\infty\}$

$$\begin{aligned}
R(\tau) &= \int_{-\infty}^{\tau} g(t)\,g(t-\tau)\,dt + \int_{\tau}^{0} g(t)\,g(t-\tau)\,dt + \int_{0}^{\infty} g(t)\,g(t-\tau)\,dt \\
&= \int_{-\infty}^{\tau} -e^{at}\left(-e^{a(t-\tau)}\right)dt + \int_{\tau}^{0} -e^{at}e^{-a(t-\tau)}dt + \int_{0}^{\infty} e^{-at}e^{-a(t-\tau)}dt \\
&= e^{-a\tau}\int_{-\infty}^{\tau} e^{2at}dt - e^{a\tau}\int_{\tau}^{0} 1dt + e^{a\tau}\int_{0}^{\infty} e^{-2at}dt \\
&= e^{-a\tau}\frac{\left[e^{2at}\right]_{-\infty}^{\tau}}{2a} + \tau e^{a\tau} + e^{a\tau}\frac{\left[e^{-2at}\right]_{0}^{\infty}}{-2a} \\
&= e^{-a\tau}\frac{\left[e^{2a\tau}-0\right]}{2a} + \tau e^{a\tau} + e^{a\tau}\frac{\left[0-1\right]}{-2a} \\
&= \frac{e^{a\tau}}{2a} + \tau e^{a\tau} + \frac{e^{a\tau}}{2a} \\
&= e^{a\tau}\left(\frac{1}{a}+\tau\right)
\end{aligned}$$

At $\tau = 0$, we see that $R(0) = \frac{1}{a}$, hence the final answer is

$$R(\tau) = \begin{cases} e^{-a\tau}\left(\frac{1}{a}-\tau\right) & \tau > 0 \\ \frac{1}{a} & \tau = 0 \\ e^{a\tau}\left(\frac{1}{a}+\tau\right) & \tau < 0 \end{cases}$$

Or we could write

$$\boxed{R(\tau) = e^{-|\tau|a}\left(\frac{1}{a} - |\tau|\right)}$$

This is a plot of $R(\tau)$, first plot is for $a = 1$ and the second for $a = 4$

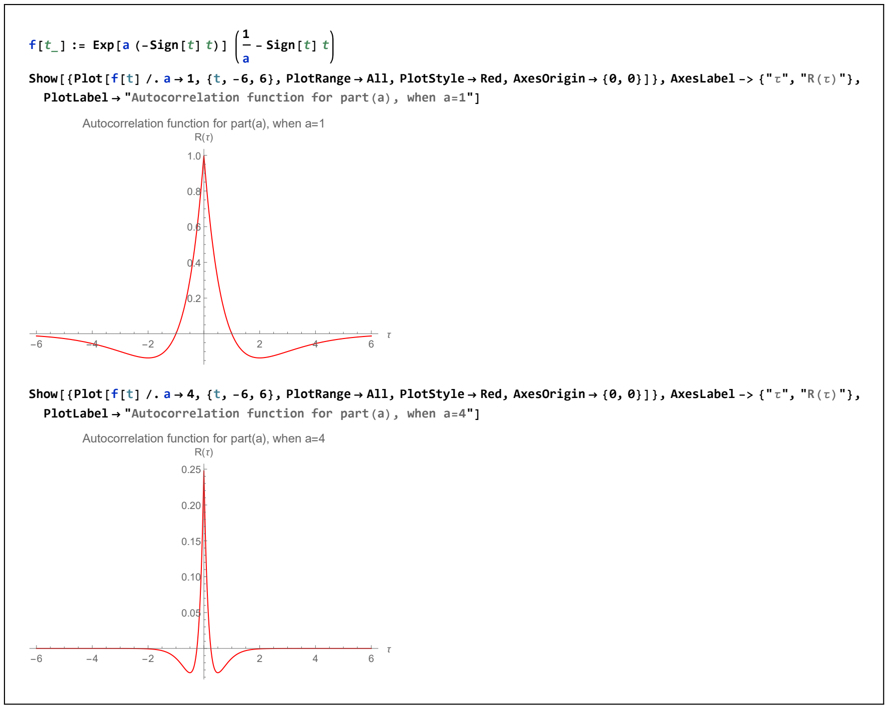



Figure 3.7: Part c

### 3.2.3 Problem 2.32

**problem**: Determine the autocorrelation function of $g(t) = A\,\mathrm{sinc}(2Wt)$ and sketch it

**solution**:

$$R(\tau) = \int_{-\infty}^{\infty} g(t)\, g^*(t-\tau)\, dt$$

The above is difficult to do directly, hence we use the second method.

Since the function $g(t)$ is an energy function, hence $R(\tau)$ and the energy spectrum density $\Psi_g(f)$ of $g(t)$make a Fourier transform pairs.

$$R(\tau) \Leftrightarrow \Psi_g(f)$$

Therefore, to find $R(\tau)$, we first find $\Psi_g(f)$, then find the Inverse Fourier Transform of $\Psi_g(f)$, i.e.

$$R(\tau) = F^{-1}(\Psi_g(f)) \tag{1}$$

But

$$\Psi_g(f) = |G(f)|^2 \tag{2}$$

and we know that

$$A\,\mathrm{sinc}(2Wt) \Leftrightarrow \frac{A}{2W} rect\left(\frac{f}{2W}\right)$$

Hence

$$G(f) = \frac{A}{2W} rect\left(\frac{f}{2W}\right)$$

The (2) becomes

$$\begin{aligned}\Psi_g(f) &= \left|\frac{A}{2W} rect\left(\frac{f}{2W}\right)\right|^2 \\ &= \left(\frac{A}{2W}\right)^2 \left|rect\left(\frac{f}{2W}\right)\right|^2\end{aligned}$$

But $\left|rect\left(\frac{f}{2W}\right)\right|^2 = rect\left(\frac{f}{2W}\right)$, since it has height of 1, so

$$\boxed{\Psi_g(f) = \left(\frac{A}{2W}\right)^2 rect\left(\frac{f}{2W}\right)}$$

Hence from (1)

$$\begin{aligned}R(\tau) &= F^{-1}\left(\left(\frac{A}{2W}\right)^2 rect\left(\frac{f}{2W}\right)\right) \\ &= \left(\frac{A}{2W}\right)^2 F^{-1}\left[rect\left(\frac{f}{2W}\right)\right]\end{aligned}$$

Hence

$$\boxed{R(\tau) = \left(\frac{A}{2W}\right)^2 \mathrm{sinc}(2W\tau)}$$

This is a plot of the above function, for $W = 4$, and $A = 1$

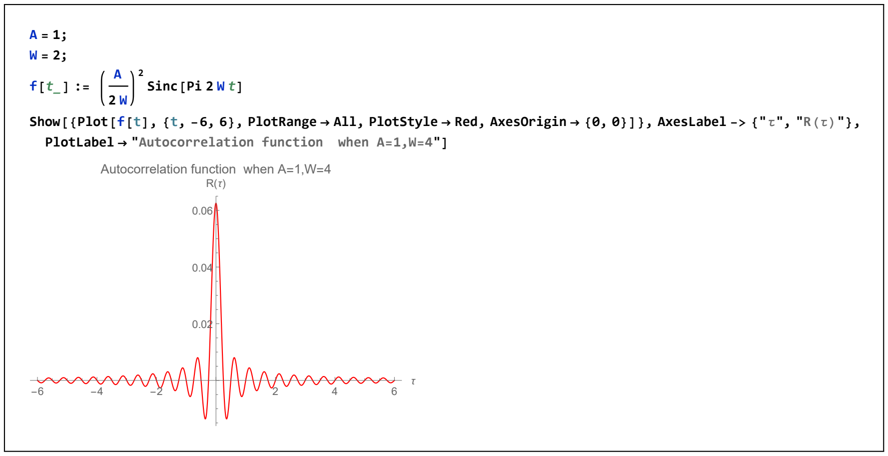



Figure 3.8: Plot for $W = 4$, and $A = 1$

### 3.2.4 Problem 2.33

The Fourier transform of a signal is defined by $|\text{sinc}(f)|$. Show that $R(\tau)$ of the signal is triangular in form.

Answer:

Since

$$R(\tau) \Leftrightarrow |G(f)|^2$$

Then

$$\begin{aligned} R(\tau) &\Leftrightarrow |\text{sinc}(f)|^2 \\ &\Leftrightarrow \text{sinc}^2(f) \end{aligned}$$

Hence to find $R(\tau)$ we need to find the inverse Fourier transform of $\text{sinc}^2(f)$

But

$$\begin{aligned} F^{-1}\left(\text{sinc}^2(f)\right) &= F^{-1}(\text{sinc}(f) \times \text{sinc}(f)) \\ &= F^{-1}\{\text{sinc}(f)\} \otimes F^{-1}\{\text{sinc}(f)\} \end{aligned}$$

But $F^{-1}\{\text{sinc}(f)\} = rect(t)$, hence

$$\begin{aligned} F^{-1}\left(\text{sinc}^2(f)\right) &= rect(t) \otimes rect(t) \\ &= \int_{-\infty}^{\infty} rect(\tau)\, rect(t-\tau)\, d\tau \end{aligned}$$

This integral has the value of $tri\left(t\right)$ (we also did this in class) Hence

$$tri\left(\tau\right)\Leftrightarrow \operatorname{sinc}^{2}\left(f\right)$$

Hence

$$R\left(\tau\right)=tri\left(\tau\right)$$

Where $tri\left(\tau\right)$ is the triangle function, defined as

$$tri\left(t\right)=\begin{cases}1-\left\vert t\right\vert & \left\vert t\right\vert <0\\ 0 & otherwise\end{cases}$$

### 3.2.5 Problem 2.35

Consider the signal $g\left(t\right)$ defined by

$$g\left(t\right)=A_{0}+A_{1}\cos\left(2\pi f_{1}t+\theta\right)+A_{2}\cos\left(2\pi f_{2}t+\theta\right)$$

(a) determine $R\left(\tau\right)$

(b) what is $R\left(0\right)$

(c) has any information been lose in obtaining $R\left(\tau\right)$?

**Answer**:

(a)

Take the Fourier transform of $g\left(t\right)$ we obtain

$$G\left(f\right)=A_{0}\delta\left(f\right)+\frac{A_{1}}{2}\left[e^{j\theta}\delta\left(f-f_{1}\right)+e^{-j\theta}\delta\left(f+f_{1}\right)\right]+\frac{A_{2}}{2}\left[e^{j\theta}\delta\left(f-f_{2}\right)+e^{-j\theta}\delta\left(f+f_{2}\right)\right]$$

Hence $\left\vert G\left(f\right)\right\vert^{2}=G\left(f\right)G^{\ast}\left(f\right)$, so we need to find $G^{\ast}\left(f\right)$

$$G^{\ast}\left(f\right)=A_{0}\delta\left(f\right)+\frac{A_{1}}{2}\left[e^{-j\theta}\delta\left(f-f_{1}\right)+e^{j\theta}\delta\left(f+f_{1}\right)\right]+\frac{A_{2}}{2}\left[e^{-j\theta}\delta\left(f-f_{2}\right)+e^{j\theta}\delta\left(f+f_{2}\right)\right]$$

So

$$G\left(f\right)G^{\ast}\left(f\right)=A_{0}^{2}\delta\left(f\right)+\frac{A_{1}^{2}}{4}\left[\delta\left(f-f_{1}\right)+\delta\left(f+f_{1}\right)\right]+\frac{A_{2}^{2}}{4}\left[\delta\left(f-f_{2}\right)+\delta\left(f+f_{2}\right)\right]$$

So

$$S_{g}\left(f\right)=A_{0}^{2}\delta\left(f\right)+\frac{A_{1}^{2}}{4}\left[\delta\left(f-f_{1}\right)+\delta\left(f+f_{1}\right)\right]+\frac{A_{2}^{2}}{4}\left[\delta\left(f-f_{2}\right)+\delta\left(f+f_{2}\right)\right]$$

So

$$
\begin{aligned}
R(\tau) &= F^{-1}\left(S_g(f)\right) \\
&= F^{-1}\left(A_0^2\delta(f)\right) + \frac{A_1^2}{4}F^{-1}\left[\delta(f-f_1)+\delta(f+f_1)\right] + \frac{A_2^2}{4}F^{-1}\left[\delta(f-f_2)+\delta(f+f_2)\right]
\end{aligned}
$$

Hence

$$
\boxed{R(\tau) = A_0^2 + \tfrac{A_1^2}{2}\cos 2\pi f_1\tau + \tfrac{A_2^2}{2}\cos 2\pi f_2\tau} \tag{1}
$$

Part (b)

$$
\begin{aligned}
R(0) &= A_0^2 + \frac{A_1^2}{2} + \frac{A_2^2}{2} \\
&= \frac{1}{2}\left(2A_0^2 + A_1^2 + A_2^2\right)
\end{aligned}
$$

part(c)

In obtaining $R(\tau)$ we have lost the phase information in the original signal as can be seen from (1) above

### 3.2.6 extra Problem

(a) find $\xi(t)\otimes\xi(t)$ where $\xi(t)$ is unit step function

(b)Find $t\xi(t)\otimes e^{at}\xi(t)$ where $a>0$

(c)find $u(t)\otimes h(t)$ where $h(t)=e^{-3t}u(t)$ and $u(t)$ is as shown

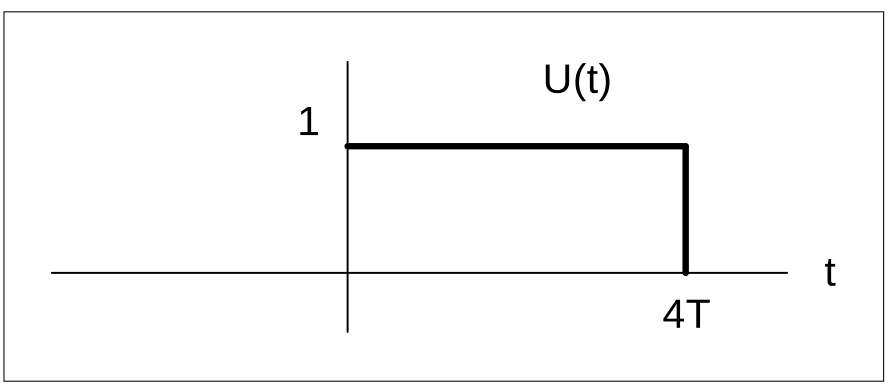



Figure 3.9: Extra problem

To DO

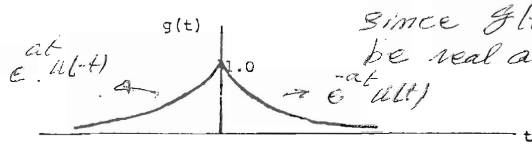
### 3.2.7 Key solution

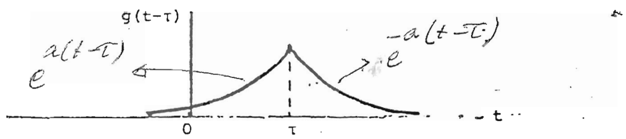
EE 443 chap 2 | HW#2 Key | page 1

1.30 a) see handout page (28)

3.0 b) $g = \exp(-a|t|) = e^{-at}u(t) + e^{at}u(-t)$ with $a>0$

since $g(t)$ is real then $R_g(\tau)$ will be real and even $\Rightarrow R_g(-\tau) = R_g(\tau)$




Therefore, for τ>0, $\Rightarrow R_g(\tau) = \int_{-\infty}^{+\infty} g(t)g(t-\tau)dt$

$$R_g(\tau) = \int_{-\infty}^{0} \exp(at)\exp[a(t-\tau)]dt$$

$$+ \int_{0}^{\tau} \exp(-at)\exp[a(t-\tau)]dt$$

$$+ \int_{\tau}^{\infty} \exp(-at)\exp[-a(t-\tau)]dt$$

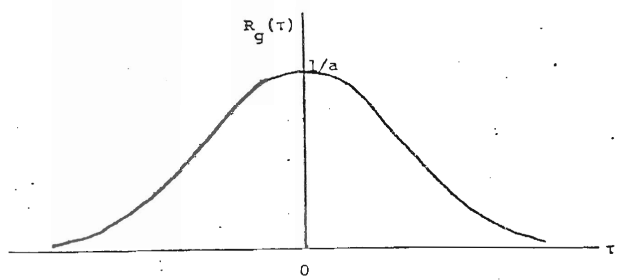
$$= \frac{1}{2a}\exp(-a\tau) + \tau\exp(-a\tau) + \frac{1}{2a}\exp(-a\tau)$$

$$= (\frac{1}{a} + \tau)\exp(-a\tau)$$

Since $R_g(-\tau) = R_g(\tau)$, we may express $R_g(\tau)$ for all τ as follows:

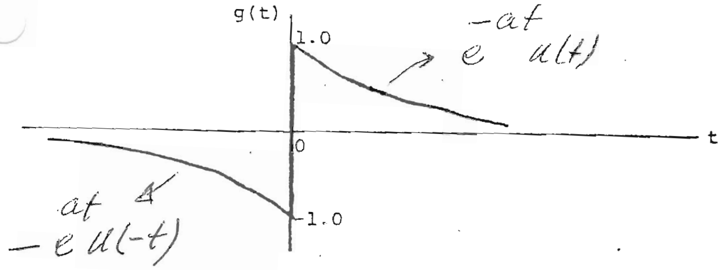
$$R_g(\tau) = (\frac{1}{a} + |\tau|)\exp(-a|\tau|)$$

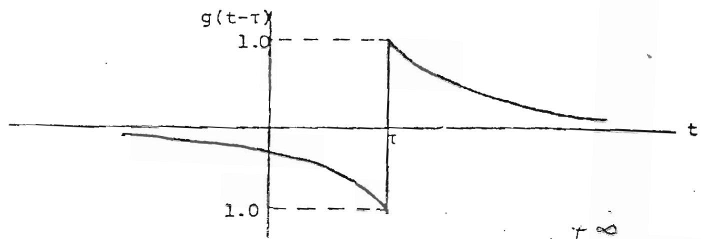
which is illustrated below:




(c) $g(t) = \exp(-at)u(t) - \exp(at)u(-t)$ , $a>0$ , $g(t)$ is real.

For τ>0, we have




Therefore, for $\tau>0$, $R_g(\tau) = \int_{-\infty}^{+\infty} g(t)g(t-\tau)dt$

$$R_g(\tau) = \int_{-\infty}^{0} \exp(at)\exp[-a(t-\tau)]dt$$

$$- \int_{0}^{\tau} \exp(-at)\exp[a(t-\tau)]dt$$

$$+ \int_{\tau}^{\infty} \exp(-at)\exp[-a(t-\tau)]dt$$

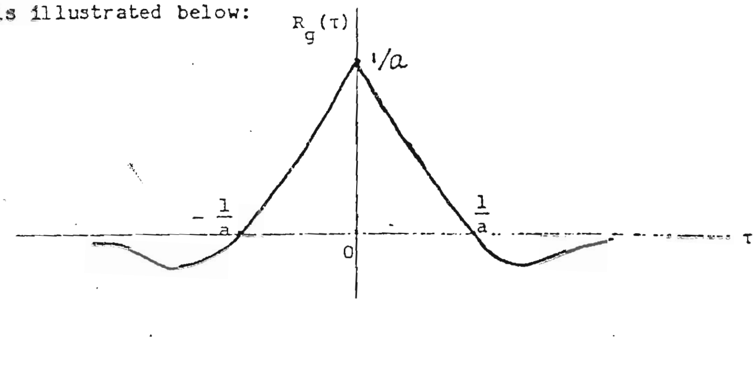
$$= \frac{1}{2a}\exp(-a\tau) - \tau\exp(-a\tau) + \frac{1}{2a}\exp(-a\tau)$$

$$= \left(\frac{1}{a} - \tau\right)\exp(-a\tau)$$

Since $R_g(-\tau) = R_g(\tau)$, we may express $R_g(\tau)$ for all $\tau$ as follows:

$$R_g(\tau) = \left(\frac{1}{a} - |\tau|\right)\exp(-a|\tau|)$$

which is illustrated below:



**2.32** $A\,\mathrm{sinc}(2Wt) \rightleftharpoons \frac{A}{2W}\,\mathrm{rect}\left(\frac{f}{2W}\right) = G(f)$

page 3

Since,

$$R_g(\tau) \rightleftharpoons |G(f)|^2,$$

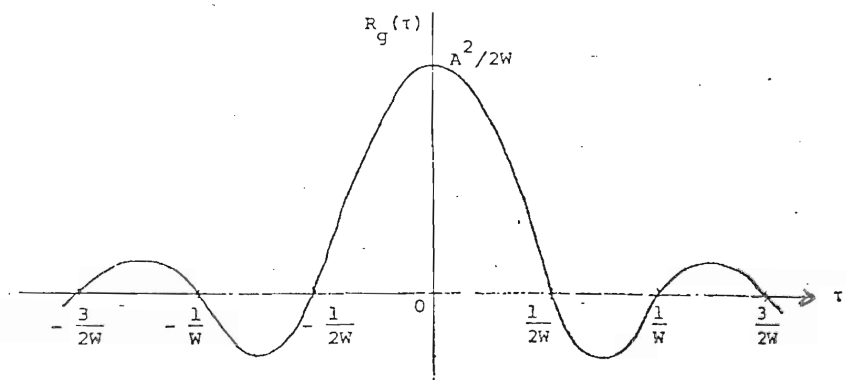
it follows that for the given sinc pulse

$$R_g(\tau) \rightleftharpoons \frac{A^2}{4W^2}\,\mathrm{rect}\left(\frac{f}{2W}\right)$$

Therefore,

$$R_g(\tau) = \frac{A^2}{2W}\,\mathrm{sinc}(2W\tau)$$

which is shown illustrated below:




Problem 2.33 → See also page (4)

$$G(f) = |\mathrm{sinc}(f)|$$

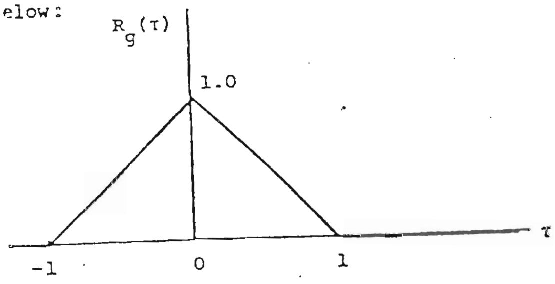
Therefore,

$$|G(f)|^2 = \mathrm{sinc}^2(f) \overset{F.T}{\longleftrightarrow} R_g(\tau)$$

The function $\mathrm{sinc}^2(f)$ represents the Fourier transform of a triangular pulse of unit amplitude and width 2 seconds, centered at the origin. Therefore,

$$R_g(\tau) = \begin{cases} 1-|\tau|, & |\tau|<1 \\ 0, & |\tau|>1 \end{cases}$$

which is illustrated below:



EE 443 | Chap 2 | HW # 2 | page 4

**2.23 :** (second method) $G(f) = |Sinc(f)|$

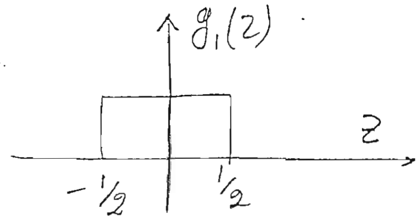
$$R_g(\tau) \overset{F.T.}{\longleftrightarrow} |G(f)|^2 = Sinc^2(f)$$

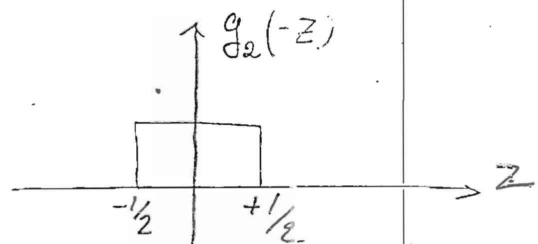
$$= \underbrace{Sinc(f)}_{G_1(f)} \cdot \underbrace{Sinc(f)}_{G_2(f)}$$

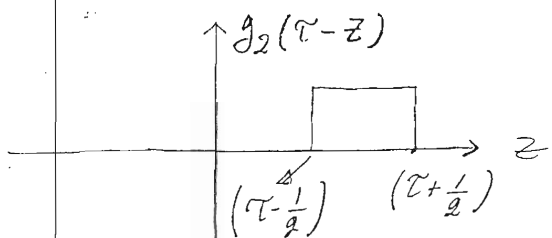
Therefore $R_g(\tau) = g_1(\tau) \otimes g_2(\tau)$

where $g_1(\tau) = g_2(\tau) = F^{-1}[Sinc(f)] = rect(\tau)$

$$R_g(\tau) = \int_{-\infty}^{+\infty} g_1(z)\, g_2(\tau - z)\, dz$$

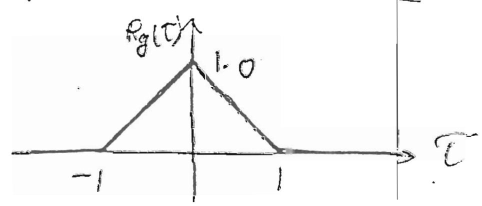



After computing the convolution we have

$$R_g(\tau) = \begin{cases} 0 & \tau < -1 \\ 1+\tau & -1 \le \tau < 0 \\ 1-\tau & 0 \le \tau < 1 \\ 0 & \tau > 1 \end{cases} \Rightarrow$$



EE 443 HW #2 page 5

2.35

(a) $g(t) = A_0 + A_1 \cos(2\pi f_1 t+\theta)+A_2 \cos(2\pi f_2 t+\theta)$

Therefore:

$$G(f) = A_0\,\delta(f) + \frac{A_1}{2}[\delta(f-f_1)\exp(j\theta)+\delta(f+f_1)\exp(-j\theta)]$$

$$+ \frac{A_2}{2}[\delta(f-f_2)\exp(j\theta)+\delta(f+f_2)\exp(-j\theta)]$$

and

$$|G(f)|^2 = A_0^2\,\delta(f) + \frac{A_1^2}{4}[\delta(f-f_1)+\delta(f+f_1)] + \frac{A_2^2}{4}[\delta(f-f_2)+\delta(f+f_2)]$$

Since $R_g(\tau) \rightleftharpoons |G(f)|^2$

it follows that

$$R_g(\tau) = A_0^2 + \frac{A_1^2}{2}\cos(2\pi f_1\tau) + \frac{A_2^2}{2}\cos(2\pi f_2\tau)$$

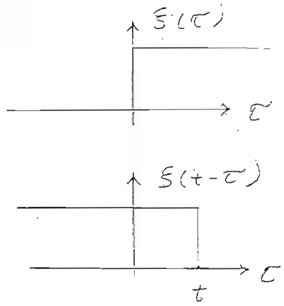
(b) $R_g(0) = A_0^2 + \frac{A_1^2}{2} + \frac{A_2^2}{2}$

(c) We see that $R_g(\tau)$ depends only on the dc component $A_0$, the amplitudes $A_1$ and $A_2$ of the two sinusoidal components and their frequencies $f_1$ and $f_2$. The phase information contained in the phase angles of the two sinusoidal components is completely lost when evaluating $R_g(\tau)$.

EE 443 | chapter | HW # | page 7

Extra prob #2) Evaluate the following convolutions:

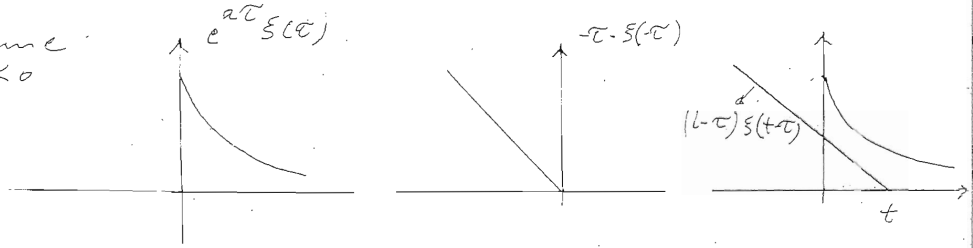
a) $\xi(t) * \xi(t) = \int_{-\infty}^{+\infty} \xi(\tau)\,\xi(t-\tau)\,d\tau$




$= \int_0^t 1 \cdot d\tau = \begin{cases} t, & \text{for } t \geq 0 \\ 0 & , \quad t < 0 \end{cases}$

b) $y(t) = t\,\xi(t) * e^{at}\,\xi(t) = \int_{-\infty}^{+\infty} e^{a\tau}\,\xi(\tau) \cdot (t-\tau)\,\xi(t-\tau)\,d\tau$

Assume $a < 0$




$y(t) = \int_0^t e^{a\tau}(t-\tau)\,d\tau = \frac{1}{a} e^{a\tau}(t-\tau)\Big|_0^t + \frac{1}{a}\int_0^t e^{a\tau}\,d\tau$

$= \frac{1}{a^2}\left(e^{at} - 1\right) - \frac{t}{a}$

c) $e^{at}\,\xi(t) * e^{at}\,\xi(t) =$

$= \int_0^t e^{a\tau} \cdot e^{a(t-\tau)}\,d\tau = \int_0^t e^{at}$

$= \begin{cases} t\,e^{at} & t \geq 0 \\ 0 & t < 0 \end{cases}$

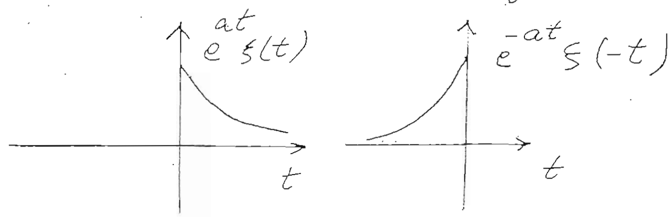
EE 433 | chapter | HW # 2 | page 2

d) Find the following convolution

(Extra problem:) $e^{at}\xi(t) * e^{-at}\xi(-t)$

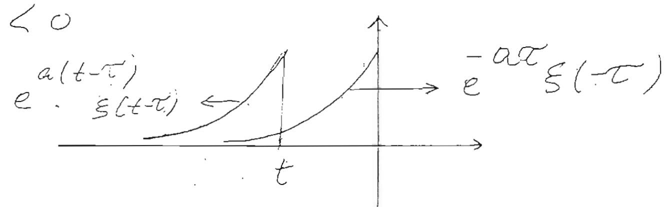
Note: The parameter $a$ must be negative otherwise the convolution integral will not converge.

$a<0$



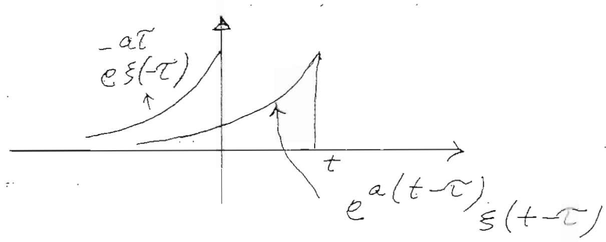

$$y(t) = e^{at}\xi(t) * e^{-at}\xi(-t) = \int_{-\infty}^{+\infty} e^{a(t-\tau)}\xi(t-\tau)\cdot e^{-a\tau}\xi(-\tau)\, d\tau$$

1) for $t<0$




$$y(t) = \int_{-\infty}^{t} e^{a(t-\tau)} e^{-a\tau}\, d\tau = \frac{-e^{-at}}{2a}$$

2) for $t \geq 0$

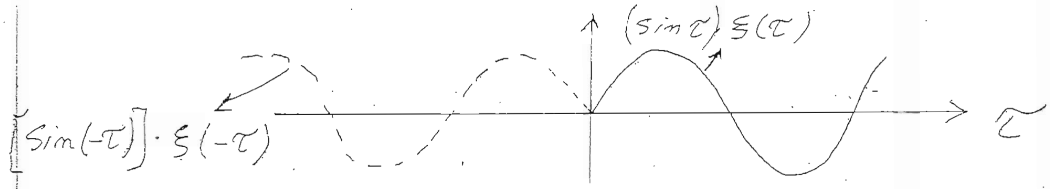



$$y(t) = \int_{-\infty}^{0} e^{a(t-\tau)} \cdot e^{-a\tau}\, d\tau = -\frac{e^{at}}{2a}$$

Thus: $y(t) = \begin{cases} -\frac{e^{-at}}{2a} & t<0 \\ -e^{at} & t\geq 0 \end{cases} \Rightarrow y(t) = -\frac{e^{a|t|}}{2a}$

EE 443 | chapter | HW # 2 | page

e) $y(t) = (\sin t)\cdot\xi(t) * \sin t\ \xi(t)$




$y(t) = 0 \quad \text{for } t \le 0$

$$y(t) = \int_0^t \sin\tau \cdot \sin(t-\tau)\, d\tau \quad \text{for } t > 0$$

$$= \frac{1}{2}\sin t - \frac{1}{2} t \cos t$$

$$y(t) = \int_0^t \frac{1}{2}\left[\cos(t-\tau-\tau) - \cos(t-\tau+\tau)\right] d\tau$$

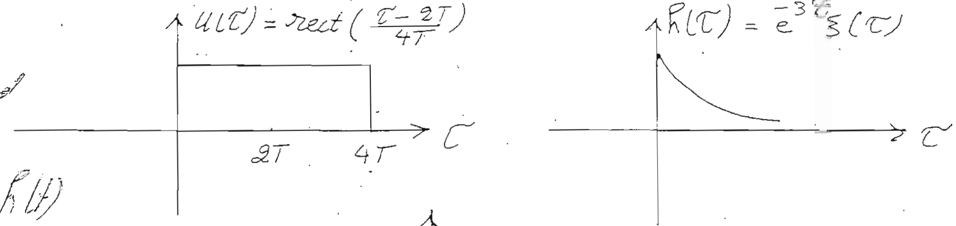
$$= \frac{1}{2}\int_0^t \left[\cos(t-2\tau) - \cos t\right] d\tau = \frac{1}{2}\left[-\frac{1}{2}\sin(t-2\tau) - \tau\cos t\right]_0^t$$

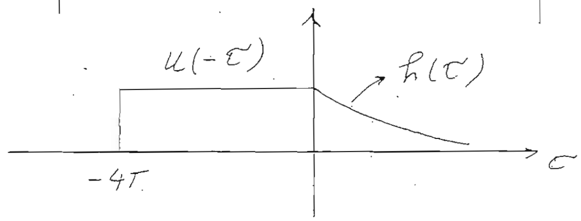
$$= \frac{1}{2}\left[-\frac{1}{2}\sin(-t) + \frac{1}{2}\sin(t) - t\cos t\right] = \frac{1}{2}\left[\sin t - t\cos t\right]$$

EE 443 | HW #2 | page 4

f)

Given $u(t)$ and $h(t)$ find $y(t) = u(t) \circledast h(t)$



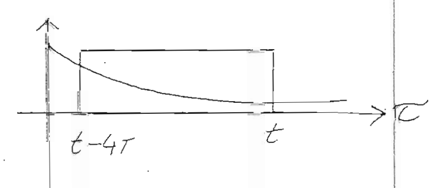




a) For $t < 0 \Rightarrow y(t) = 0$

b) For $0 \leq t < 4T$

$$y(t) = \int_0^t 1 \cdot e^{-3\tau}\, d\tau = -\left.\frac{e^{-3\tau}}{3}\right|_0^t = \frac{1 - e^{-3t}}{3}$$

c) For $t \geq 4T$




$$y(t) = \int_{t-4T}^{t} e^{-3\tau}\, d\tau = \frac{1}{3}\left(e^{-3t+12T} - e^{-3t}\right)$$

$$= \frac{e^{-3t}}{3}\left(e^{12T} - 1\right)$$

From a, b, c we have

## 3.3 HW 3

### Local contents

### 3.3.1 questions

443 HW #3 Sept. 17, 2008.

CH. 2 / SIGNALS AND SPECTRA

2-3 The voltage across a load is given by $v(t) = A_0 \cos \omega_0 t$, and the current through the load is a square wave,

$$i(t) = I_0 \sum_{n=-\infty}^{\infty} \left[ \Pi\left(\frac{t - nT_0}{T_0/2}\right) - \Pi\left(\frac{t - nT_0 - (T_0/2)}{T_0/2}\right) \right]$$

where $\omega_0 = 2\pi/T_0$, $T_0 = 1$ sec, $A_0 = 10$ V, and $I_0 = 5$ mA.

(a) Find the expression for the instantaneous power and sketch this result as a function of time.

(b) Find the value of the average power.

2-4 The voltage across a 50-Ω resistive load is the positive portion of a cosine wave That is,

$$v(t) = \begin{cases} 10 \cos \omega_0 t, & |t - nT_0| < T_0/4 \\ 0, & t \text{ elsewhere} \end{cases}$$

where $n$ is any integer.

(a) Sketch the voltage and current waveforms.

(b) Evaluate the dc values for the voltage and current.

(c) Find the rms values for the voltage and current.

(d) Find the total average power dissipated in the load.

2-5 For Prob. 2-4, find the energy dissipated in the load during a 1-hr interval if $T_0$ = 1 sec.

2-6 Determine whether each of the following signals is an energy signal or a pow signal and evaluate the normalized energy or power, as appropriate.

(a) $w(t) = \Pi(t/T_0)$.

(b) $w(t) = \Pi(t/T_0) \cos \omega_0 t$.

(c) $w(t) = \cos^2 \omega_0 t$.

✓ 2-7 An average reading power meter is connected to the output circuit of a transmitte The transmitter output is fed into a 75-Ω resistive load and the wattmeter rea 67 W.

(a) What is the power in dBm units?

(b) What is the power in dBk units?

(c) What is the value in dBmV units?

✓ 2-8 Assume that a waveform with a known rms value, $V_{rms}$, is applied across a 50- load. Derive a formula that can be used to compute the dBm value from $V_{rms}$.

✓ 2-9 An amplifier is connected to a 50-Ω load and driven by a sinusoidal current sour as shown in Fig. P2-9. The output resistance of the amplifier is 10 Ω and the inp resistance is 2 kΩ. Evaluate the true decibel gain of this circuit.

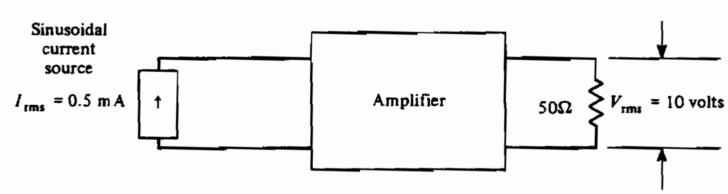



FIGURE P2-9

2-10 The voltage (rms) across the 300-$\Omega$ antenna input terminals of an FM receiver is 3.5 $\mu$V.
(a) Find the input power (watts).
(b) Evaluate the input power as measured in decibels below 1 mW (dBm).
(c) What would be the input voltage (in microvolts) for the same input power if the input resistance were 75 $\Omega$ instead of 300 $\Omega$?

2-11 What is the value for the phasor that corresponds to the voltage waveform $v(t) = 12\sin(\omega_0 t - 25°)$, where $\omega_0 = 2000\pi$?

2-12 A signal is $w(t) = 3\sin(100\pi t - 30°) + 4\cos(100\pi t)$. Find the corresponding phasor.

2-13 Evaluate the Fourier transform of

$$w(t) = \begin{cases} e^{-\alpha t}, & t \geq 1 \\ 0, & t < 1 \end{cases}$$

2-14 Find the spectrum for the waveform $w(t) = e^{-\pi(t/T)^2}$. What can we say about the width of $w(t)$ and $W(f)$ as $T$ increases? [*Hint:* Use (A-75).]

✓ 2-15 Using the convolution property, find the spectrum for

$$w(t) = \sin 2\pi f_1 t \cos 2\pi f_2 t$$

2-16 Find the spectrum (Fourier transform) of the triangle waveform

$$s(t) = \begin{cases} At, & 0 < t < T_0 \\ 0, & t \text{ elsewhere} \end{cases}$$

in terms of $A$ and $T_0$.

✓ 2-17 Find the spectrum for the waveform shown in Fig. P2-17.

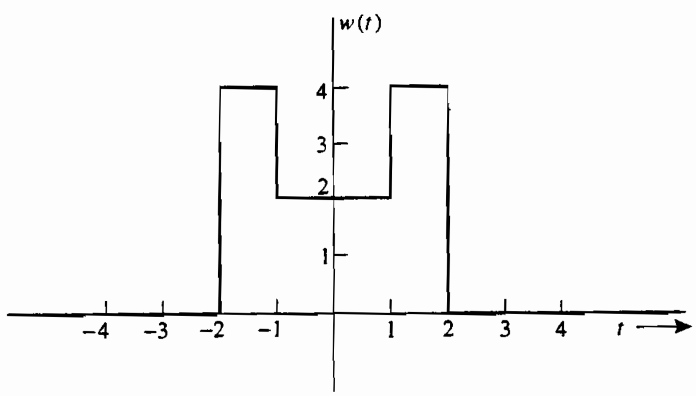



**FIGURE P2-17**

✓ 2-18 If $w(t)$ has the Fourier transform

$$W(f) = \frac{j2\pi f}{1 + j2\pi f}$$

find $X(f)$ for the following waveforms.
(a) $x(t) = w(2t + 2)$.

(b) $x(t) = w(t-1)e^{-jt}$

c) $x(t) = w(1-t)$

## 3.3.2 Problem 2.7

**Problem** An average reading power meter is connected to output of transmitter. Transmitter output is fed into $75\Omega$ resistive load and the wattmeter read $67W$

(a) What is power in dBm units?

(b) What is power in dBk units?

(c) What is the value in dBmV units?

### 3.3.2.1 part(a)

$$\begin{aligned}P_{dbm} &= 10\log_{10} P_m \\ &= 10\log_{10}(67000) \\ &= \boxed{48.2607}\ dbm\end{aligned}$$

(b)

$$\begin{aligned}P_{dbk} &= 10\log_{10} P_k \\ &= 10\log_{10}(0.067) \\ &= \boxed{-11.7393}\ dbk\end{aligned}$$

(c)

$$P = \frac{V^2}{R}$$

Hence

$$10\log_{10} P = 20\log_{10} V - 10\log_{10} R$$

Hence

$$20\log_{10} V = 10\log_{10} P + 10\log_{10} R$$

so

$$\begin{aligned}20\log_{10} V &= 10\log_{10} 67000 + 10\log_{10} 75000 \\ &= \boxed{97.0114 \text{ dbmV}}\end{aligned}$$

### 3.3.3 Problem 2.8

Assume that a waveform with known rms value $V_{rms}$ is applied across a $50\Omega$ load. Derive a formula that can be used to computer the *dbm* value from $V_{rms}$

$$P\,(watt) = \frac{V_{rms}^2\,(V)}{R\,(\Omega)}$$

Hence

$$\begin{aligned}
P_{dbm} &= 10\log_{10}\left(10^3 \times P_{watt}\right) \\
&= 10\log_{10}\frac{10^3 \times V_{rms}^2\,(V)}{R\,(\Omega)} \\
&= 10\left(\log_{10} 10^3 V_{rms}^2 - \log_{10} R\right) \\
&= 10\left(\log_{10} 10^3 + \log_{10} V_{rms}^2 - \log_{10} R\right) \\
&= 10\,(3 + 2\log_{10} V_{rms} - \log_{10} R)
\end{aligned}$$

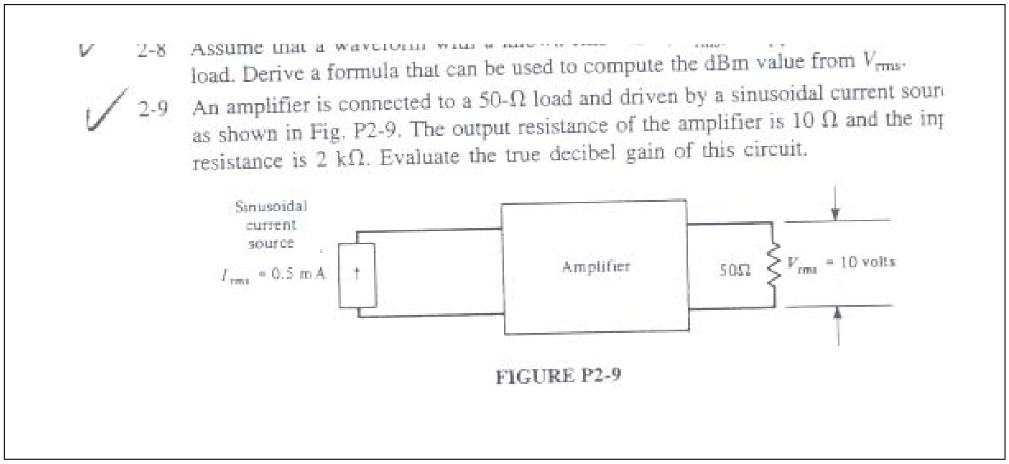
Hence

$$\boxed{P_{dbm} = 30 + 20\log_{10} V_{rms} - 10\log_{10} R}$$

When $R = 50\Omega$, we obtain

$$\begin{aligned}
P_{dbm} &= 30 + 20\log_{10} V_{rms} - 10\log_{10} 50 \\
&= 30 + 20\log_{10} V_{rms} - 16.9897 \\
&= 13.0103 + 20\log_{10} V_{rms}
\end{aligned}$$

### 3.3.4 Problem 2.9


2-8 Assume that a waveform with a ... load. Derive a formula that can be used to compute the dBm value from $V_{rms}$.

2-9 An amplifier is connected to a 50-Ω load and driven by a sinusoidal current soun as shown in Fig. P2-9. The output resistance of the amplifier is 10 Ω and the inp resistance is 2 kΩ. Evaluate the true decibel gain of this circuit.


FIGURE P2-9

Figure 3.10: the Problem statement

$$
\begin{aligned}
Gain(db) &= 10\log_{10}\frac{P_L}{P_i}\\
&= 10\log_{10}\frac{\left(\frac{V_{rms}^2}{R_L}\right)}{I_{rms}^2 R_{in}}\\
&= 10\log_{10}\frac{\left(\frac{10^2}{50}\right)}{\left(0.5\times10^{-3}\right)^2\times2000}\\
&= 10\log_{10}\frac{10^5}{25}\\
&= 10\left(\log_{10}10^5-\log_{10}25\right)\\
&= 10\left(5-1.39794\right)\\
&= 36.021
\end{aligned}
$$

### 3.3.5 Problem 2.15

Using the convolution property find the spectrum for $w(t)=\sin 2\pi f_1 t\ \cos 2\pi f_2 t$

Solution:

$$
\mathcal{F}\left(w\left(t\right)\right)=\mathcal{F}\left(\sin 2\pi f_1 t\right)\otimes\mathcal{F}\left(\cos 2\pi f_2 t\right) \tag{1}
$$

But

$$
\begin{aligned}
\mathcal{F}\left(\sin 2\pi f_1 t\right) &= \frac{1}{2j}\left(\delta\left(f-f_1\right)-\delta\left(f+f_1\right)\right)\\
\mathcal{F}\left(\cos 2\pi f_2 t\right) &= \frac{1}{2}\left(\delta\left(f-f_2\right)+\delta\left(f+f_2\right)\right)
\end{aligned}
$$

Hence (1) becomes

$$\begin{aligned} \digamma(w(t)) &= \left\{\frac{1}{2j}\left(\delta(f-f_1)-\delta(f+f_1)\right)\right\} \otimes \left\{\frac{1}{2}\left(\delta(f-f_2)+\delta(f+f_2)\right)\right\} \\ &= \frac{1}{4j}\left\{\delta(f-f_1)-\delta(f+f_1)\right\} \otimes \left\{\delta(f-f_2)+\delta(f+f_2)\right\} \end{aligned} \tag{2}$$

Applying the distributed property of convolution, i.e. $a\otimes(b+c)=a\otimes b+a\otimes c$ on equation (2) we obtain

$$4j\,\digamma(w(t)) = \delta(f-f_1)\otimes\delta(f-f_2)+\delta(f-f_1)\otimes\delta(f+f_2)-\delta(f+f_1)\otimes\delta(f-f_2)-\delta(f+f_1)\otimes\delta(f+f_2) \tag{3}$$

Now

$$\begin{aligned} \delta(f-f_1)\otimes\delta(f-f_2) &= \int_{-\infty}^{\infty}\delta(\lambda-f_1)\,\delta(f-(\lambda-f_2))\,d\lambda \\ &= \delta(f+f_2-f_1)\int_{-\infty}^{\infty}\delta(f-(\lambda-f_2))\,d\lambda \\ &= \delta(f+f_2-f_1) \end{aligned} \tag{4}$$

And

$$\begin{aligned} \delta(f-f_1)\otimes\delta(f+f_2) &= \int_{-\infty}^{\infty}\delta(\lambda-f_1)\,\delta(f-(\lambda+f_2))\,d\lambda \\ &= \delta(f-f_2-f_1)\int_{-\infty}^{\infty}\delta(f-(\lambda-f_2))\,d\lambda \\ &= \delta(f-f_2-f_1) \end{aligned} \tag{5}$$

And

$$\begin{aligned} \delta(f+f_1)\otimes\delta(f-f_2) &= \int_{-\infty}^{\infty}\delta(\lambda+f_1)\,\delta(f-(\lambda-f_2))\,d\lambda \\ &= \delta(f+f_2+f_1)\int_{-\infty}^{\infty}\delta(f-(\lambda-f_2))\,d\lambda \\ &= \delta(f+f_2+f_1) \end{aligned} \tag{6}$$

And

$$\begin{aligned} \delta(f+f_1)\otimes\delta(f+f_2) &= \int_{-\infty}^{\infty}\delta(\lambda+f_1)\,\delta(f-(\lambda+f_2))\,d\lambda \\ &= \delta(f-f_2+f_1)\int_{-\infty}^{\infty}\delta(f-(\lambda-f_2))\,d\lambda \\ &= \delta(f-f_2+f_1) \end{aligned} \tag{7}$$

Substitute (4,5,6,7) into (3) we obtain

$$\boxed{F\left(w\left(t\right)\right)=\frac{1}{4j}\left[\delta\left(f+f_{2}-f_{1}\right)+\delta\left(f-f_{2}-f_{1}\right)-\delta\left(f+f_{2}+f_{1}\right)-\delta\left(f-f_{2}+f_{1}\right)\right]}$$

or

$$\boxed{F\left(w\left(t\right)\right)=\frac{1}{4j}\left[\delta\left(f+\left(f_{2}-f_{1}\right)\right)+\delta\left(f-\left(f_{2}+f_{1}\right)\right)-\delta\left(f+\left(f_{2}+f_{1}\right)\right)-\delta\left(f-\left(f_{2}-f_{1}\right)\right)\right]} \tag{8}$$

This problem can also be solved as follows

$$w\left(t\right)=\sin 2\pi f_{1}t\ \ \cos 2\pi f_{2}t$$

Using $\sin\alpha\cos\beta=\frac{1}{2}\left(\sin\left(\alpha-\beta\right)+\sin\left(\alpha+\beta\right)\right)$, hence

$$\begin{aligned}
w\left(t\right) &= \frac{1}{2}\left(\sin\left(2\pi f_{1}t-2\pi f_{2}t\right)+\sin\left(2\pi f_{1}t+2\pi f_{2}t\right)\right)\\
&= \frac{1}{2}\left(\sin\left(2\pi\left(f_{1}-f_{2}\right)t\right)+\sin\left(2\pi\left(f_{1}+f_{2}\right)t\right)\right)\\
&= \frac{1}{2}\left(\frac{1}{2j}\left(\delta\left(f-\left(f_{1}-f_{2}\right)\right)-\delta\left(f+\left(f_{1}-f_{2}\right)\right)\right)+\frac{1}{2j}\left(\delta\left(f-\left(f_{1}+f_{2}\right)\right)-\delta\left(f+\left(f_{1}+f_{2}\right)\right)\right)\right)\\
&= \frac{1}{4j}\left\{\delta\left(f-\left(f_{1}-f_{2}\right)\right)-\delta\left(f+\left(f_{1}-f_{2}\right)\right)+\delta\left(f-\left(f_{1}+f_{2}\right)\right)-\delta\left(f+\left(f_{1}+f_{2}\right)\right)\right\}\\
&= \frac{1}{4j}\left\{\delta\left(f+\left(f_{2}-f_{1}\right)\right)+\delta\left(f-\left(f_{2}+f_{1}\right)\right)-\delta\left(f+\left(f_{2}+f_{1}\right)\right)-\delta\left(f-\left(f_{2}-f_{1}\right)\right)\right\}
\end{aligned} \tag{9}$$

Compare (8) and (9) we see they are the same.

### 3.3.6 Problem 2.17

$$w\left(t\right)=4\ rect\left(\frac{t}{4}\right)-2\ rect\left(\frac{t}{2}\right)$$

By linearity of Fourier Transform

$$F\left(w\left(t\right)\right)=4\times F\left(rect\left(\frac{t}{4}\right)\right)-2\times F\left(rect\left(\frac{t}{2}\right)\right) \tag{1}$$

Since

$$F\left(rect\left(\frac{t}{4}\right)\right)=4\operatorname{sinc}\left(4f\right)$$

and

$$\digamma\left(rect\left(\frac{t}{2}\right)\right) = 2\,\mathrm{sinc}\,(2f)$$

Then (1) becomes

$$\begin{aligned}\digamma\,(w\,(t)) &= 4\times 4\,\mathrm{sinc}\,(4f) - 2\times 2\,\mathrm{sinc}\,(2f)\\ &= \boxed{16\,\mathrm{sinc}\,(4f) - 4\,\mathrm{sinc}\,(2f)}\end{aligned}$$

Or in terms of just the sin function, the above becomes

$$\begin{aligned}\digamma\,(w\,(t)) &= 16\frac{\sin(4\pi f)}{4\pi f} - 4\frac{\sin(2\pi f)}{2\pi f}\\ &= 4\frac{\sin(4\pi f)}{\pi f} - 2\frac{\sin(2\pi f)}{\pi f}\\ &= \boxed{\tfrac{4\sin(4\pi f)-2\sin(2\pi f)}{\pi f}}\end{aligned}$$

## 3.3.7 Problem 2.18

If $w\,(t)$ has the Fourier Transform $W\,(f) = \frac{j2\pi f}{1+j2\pi f}$ find $X\,(f)$ for the following waveforms

(a) $x\,(t) = w\,(2t+2)$

(b) $x\,(t) = w\,(t-1)\,e^{-jt}$

(c) $x\,(t) = w\,(1-t)$

Answer:

### 3.3.7.1 Part(a)

$$w\,(t) \Leftrightarrow \frac{j2\pi f}{1+j2\pi f}$$

Then

$$\begin{aligned}w\,(2t) &\Leftrightarrow \frac{1}{2}X\left(\frac{f}{2}\right)\\ w\,(2t+2) &\Leftrightarrow \frac{1}{2}X\left(\frac{f}{2}\right)e^{j2\pi\frac{f}{2}(2)}\end{aligned}$$

Hence

$$\begin{aligned}w\,(2t+2) &\Leftrightarrow \frac{1}{2}\left(\frac{j2\pi\frac{f}{2}}{1+j2\pi\frac{f}{2}}\right)e^{j2\pi f}\\ &\Leftrightarrow \frac{1}{2}\left(\frac{j\pi f}{1+j\pi f}\right)e^{j2\pi f}\end{aligned}$$

This can be simplified to

$$\boxed{w\,(2t+2) \Leftrightarrow \tfrac{\pi f}{2(\pi f - j)}e^{j2\pi f}}$$

#### 3.3.7.2 Part(b)

$$w(t) \Leftrightarrow \frac{j2\pi f}{1+j2\pi f}$$

$$w(t-1) \Leftrightarrow X(f)\, e^{-j2\pi f(-1)}$$
$$w(t-1) \Leftrightarrow X(f)\, e^{j2\pi f}$$

Now Let $e^{-jt} = e^{-j2\pi f_0 t}$, hence $2\pi f_0 = 1$ or $f_0 = \frac{1}{2\pi}$, then

$$w(t-1)\, e^{-j2\pi f_0 t} \Leftrightarrow X(f+f_0)\, e^{j2\pi(f+f_0)}$$

Hence

$$w(t-1)\, e^{-jt} \Leftrightarrow \frac{j2\pi(f+f_0)}{1+j2\pi(f+f_0)} e^{j2\pi(f+f_0)}$$
$$w(t-1)\, e^{-jt} \Leftrightarrow \frac{j2\pi\left(f+\frac{1}{2\pi}\right)}{1+j2\pi\left(f+\frac{1}{2\pi}\right)} e^{j2\pi\left(f+\frac{1}{2\pi}\right)}$$
$$w(t-1)\, e^{-jt} \Leftrightarrow \frac{j2\pi(2\pi f+1)}{2\pi+j2\pi(2\pi f+1)} e^{j(2\pi f+1)}$$
$$w(t-1)\, e^{-jt} \Leftrightarrow \frac{j4\pi^2 f+j2\pi}{2\pi+j4\pi^2 f+j2\pi} e^{j2\pi f} e^{j}$$
$$w(t-1)\, e^{-jt} \Leftrightarrow \frac{2\pi f+1}{-j+2\pi f+1} e^{j2\pi f} e^{j}$$

Hence

$$\boxed{w(t-1)\, e^{-jt} \Leftrightarrow \frac{2\pi f+1}{1-j+2\pi f} e^{j(2\pi f+1)}}$$

#### 3.3.7.3 Part(c)

$$w(t) \Leftrightarrow \frac{j2\pi f}{1+j2\pi f}$$
$$w(-t) \Leftrightarrow X(-f)$$

Then

$$w(-t+1) \Leftrightarrow X(-f)\, e^{j2\pi f(1)}$$
$$w(1-t) \Leftrightarrow \frac{-j2\pi f}{1-j2\pi f} e^{j2\pi f}$$

### 3.3.8 Key solution

EE 443 HW # 3 page 1

Key Solution.

**2-7.**

(a) $dBm = 10\log\left(\frac{P_w}{0.001}\right) = 10\log\left(\frac{67}{0.001}\right) = \underline{48.26\,dBm}$

(b) $dBk = 10\log\left(\frac{P_w}{1000}\right) = 10\log\left(\frac{67}{1000}\right) = \underline{-11.74\,dBk}$

(c) $P = \frac{V_{rms}^2}{R}$

$\Rightarrow V_{rms} = \sqrt{PR} = \sqrt{(67)(75)} = 70.9$ volts

$\Rightarrow dBV = 20\log\left(\frac{70.9}{10^{-3}}\right) = \underline{97\,dBmV}$

**2-8.** $P = \frac{V_{rms}^2}{R} = \frac{V_{rms}^2}{50}$

$dBm = 10\log_{10}\left(\frac{P}{0.001}\right) = 10\log_{10}\left(\frac{V_{rms}^2}{0.050}\right) = 20\log_{10}(V_{rms}) - 10\log_{10}(0.050)$

$\Rightarrow \underline{dBm = 20\log_{10}(V_{rms}) + 13}$

**2-9.** $P_{in} = I_{rms}^2 R_{in} = (0.5\times10^{-3})^2(2\times10^3) = 5.0\times10^{-4}\,W$

$P_{out} = \frac{V_{rms}^2}{R_{Load}} = \frac{100}{50} = 2\,W$

$dB = 10\log_{10}\left(\frac{P_{out}}{P_{in}}\right) = 10\log\left(\frac{2}{5.0\times10^{-4}}\right) = \underline{36\ dB}$

**2-15.** $w(t) = \sin(2\pi f_1 t)\cos(2\pi f_2 t) = w_1(t)\,w_2(t)$

$\Rightarrow W(f) = W_1(f) * W_2(f) = \left[\frac{j}{2}\delta(f+f_1) - \frac{j}{2}\delta(f-f_1)\right] * \left[\frac{1}{2}\delta(f+f_2) + \frac{1}{2}\delta(f-f_2)\right]$

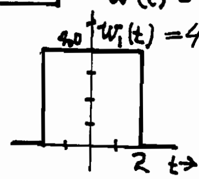
Aside: $\delta(f+f_1) * \delta(f+f_2) = \int_{-\infty}^{\infty}\delta(\lambda+f_1)\,\delta(f-\lambda+f_2)\,d\lambda = \delta(f+f_1+f_2)$

thus $\underline{W(f) = (j/4)\left[\delta(f+f_1+f_2) + \delta(f+f_1-f_2) - \delta(f-f_1+f_2) - \delta(f-f_1-f_2)\right]}$

EE 443 Hw #3 page 2

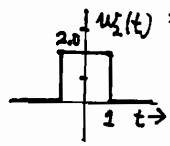
**2-17.** $w(t) = w_1(t) - w_2(t)$ where

$w_1(t) = 4\,\mathrm{rect}\left(\frac{t}{4}\right)$




$$\longleftrightarrow W_1(f) = 16\,\mathrm{sinc}(4f)$$

$w_2(t) = 2\,\mathrm{rect}\left(\frac{t}{2}\right)$




$$\longleftrightarrow W_2(f) = 4\,\mathrm{sinc}(2f)$$

$$\Rightarrow W(f) = W_1(f) - W_2(f) = 16\,\mathrm{sinc}(4f) - 4\,\mathrm{sinc}(2f)$$

**2-18.**

(a) $w(2t) \longleftrightarrow \frac{1}{2}\frac{j\pi f}{1+j\pi f}$

$$\Rightarrow x(t) = w(2(t+1)) \longleftrightarrow \frac{j\pi f}{2(1+j\pi f)} e^{j2\pi f}$$

(b) $w(t-1) \longleftrightarrow \frac{j2\pi f}{1+j2\pi f} e^{-j2\pi f}$

$$x(t) = e^{-jt} w(t-1) \longleftrightarrow \frac{j2\pi(f+\frac{1}{2\pi})}{1+j2\pi(f+\frac{1}{2\pi})} e^{-j2\pi(f+\frac{1}{2\pi})}$$

(c) $2\frac{dw(t)}{dt} \longleftrightarrow 2\left(j2\pi f\, W(f)\right)$

$$\Rightarrow x(t) \longleftrightarrow j4\pi f\left[\frac{j2\pi f}{1+j2\pi f}\right] = -\frac{8\pi^2 f^2}{1+j2\pi f}$$

(d) $w(-t) \longleftrightarrow W(-f) = -\frac{j2\pi f}{1-j2\pi f}$

$$\Rightarrow x(t) = w(-(t-1)) \longleftrightarrow \frac{-j2\pi f}{1-j2\pi f} e^{-j2\pi f}$$

EE 443 HW # 2 (Hint) page 1

2.17) $P = 67\ W$ and $R = 50\ \Omega$

a,b) $P_{dBW} \triangleq 10 \log_{10} P(w)$

$P_{dBm} = 10 \log_{10} P(mw)$ , $P_{dBK} = 10 \log_{10} P(KW)$

c) • For normalized case ($R = 1\ \Omega$), the average power P and rms voltage are related:

$P_{av} = V_{rms}^2 = \frac{V_{peak}^2}{2} \Rightarrow V_{rms} = \frac{V_{peak}}{\sqrt{2}}$

• For not normalized case ($R \neq 1\ \Omega$)

$P_{av} = \frac{V_{rms}^2}{R} = \frac{V_{peak}^2}{2R}$

if P and R are given => find $V_{rms}$.

$V_{rms}$ in dBmV is : $V_{rms}(dBmV) \triangleq 20 \log_{10} V_{rms}(mV)$

2.8) Given a sine wave like:

$v(t) = V_{peak} \cos w_0 t$ (normalized)

the average power of this periodic wave over one period, $T_0 = \frac{1}{f_0}$, is:

$$P_{av} \triangleq \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} v^2(t)\,dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} V_{peak}^2 \cos^2 w_0 t\,dt$$

$$= \frac{V_{pek}^2}{2T_0} \int_{-T_0/2}^{T_0/2} (1 + \cos 2 w_0 t)\,dt = \frac{V_{pek}^2}{2}$$

If $R \neq 1\ \Omega \Rightarrow P_{av} = \frac{V_{peak}^2}{2R} = \frac{V_{rm}^2}{R}$ watts

$\Rightarrow P_{dBm} \triangleq 10 \log_{10} P(mw) = 10 \log \left[ \frac{V_{rms}^2 \times 10^3}{R} \right] = \ldots$

EE 443 | Hw #3. (Hint) | page 2

2.9) The power gain is:

$A_{p\,dB} \triangleq 10 \log \frac{P_L}{P_{in}}$ where, $P_L = \frac{V_{o(rms)}^2}{R_L}$

$P_L$ is the power transfered to the load and $P_{in}$ is the input power supplied by the source

2.17)
w(t) may be expressed in two different way:

a) $w(t) = 4\,\text{rect}\left(\frac{t}{4}\right) - 2\,\text{rect}\left(\frac{t}{2}\right)$ or

b) $w(t) = 2\,\text{rect}\left(\frac{t}{2}\right) + 4\,\text{rect}\left(\frac{t+3/2}{1}\right) + 4\,\text{rect}\left(\frac{t-3/2}{1}\right)$

Find W(f)! The two answers should be the same. If you use the second method, you may after takin F.T use $\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right]$

2.18)
if w(t) has F.T, which is $W(f) = \frac{j2\pi f}{1+j2\pi f}$

a) Find the F.T of $x(t) = w(2t+2) = w(2(t+1))$

$$w(2t) \longleftrightarrow \frac{1}{2}\cdot\frac{j2\pi\left(\frac{f}{2}\right)}{1+j2\pi\left(\frac{f}{2}\right)} \quad \text{Scaling}$$

$$x(t) = w(2(t+1)) \longleftrightarrow \frac{j\pi f}{2(1+j\pi f)}\,e^{j2\pi f\cdot 1} \quad \text{time domain shifting!}$$

## 3.4 HW 4

### Local contents

### 3.4.1 questions and hints

EE 443 HW # 4 page 1

*Problem 1* Evaluate the transfer function of a linear system represented by the block diagram shown in Fig. P2.14.

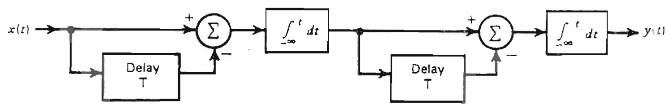



Figure P2.14

*Problem 2.*

(a) Determine the overall amplitude response of the cascade connection shown in P2.15, consisting of $N$ identical stages, each with a time constant $RC$ equal to $\tau_0$.

(b) Show that as $N$ approaches infinity, the amplitude response of the cascade connection approaches the Gaussian function $\exp(-\frac{1}{2}f^2T^2)$, where for each value of $N$, the time constant $\tau_0$ is selected so that

$$\tau_0^2 = \frac{T^2}{4\pi^2 N}$$

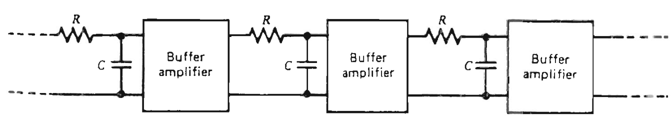



Figure P2.15

*Problem 3.* Determine the pre-envelope $g_+(t)$ corresponding to each of the following two signals:

(a) $g(t) = \text{sinc}(t)$ (Hint use $\frac{\sin t}{t} \overset{H.T.}{\longleftrightarrow} \frac{1-\cos t}{t}$ See prob # 4)

(b) $g(t) = [1 + k\cos(2\pi f_m t)]\cos(2\pi f_c t)$

prob # 4) Verify the following H.T:

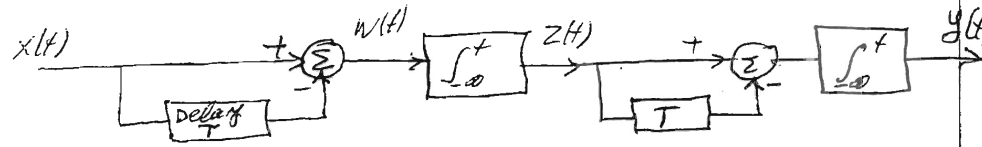
a) if $g(t) = \delta(t) \implies \hat{g}(t) = ?$

b) if $g(t) = \frac{\sin t}{t} \implies \hat{g}(t) = \frac{1-\cos t}{t}$

prob # 5) Respond prob. 2.44 of your book

EE 443 | HW # 4 (Hint) | page 1

Prob #1




Method # 1

Consider the first half the system. The second half is identical to the first half. => If $H_1(f) = \frac{Z(f)}{X(f)}$,

then $H(f) = \frac{Y(f)}{X(f)} = H_1(f) \times H_1(f) = H_1^2(f)$

- write time domain equations, then take their F.T.

$w(t) = x(t) - x(t-T) \Rightarrow W(f) = X(f)\left[1 - e^{-j2\pi fT}\right]$ (1)

$z(t) = \int_{-\infty}^{t} w(t_1)dt_1 \Rightarrow Z(f) = \frac{W(f)}{j2\pi f} + \frac{1}{2}W(0)\delta(f)$ (2)

Find $W(0)$, Combine eq. (1) and (2) to find $H_1(f) = \frac{Z(f)}{X(f)}$

Method # 2

For first half find its impulse response $h_1(t)$

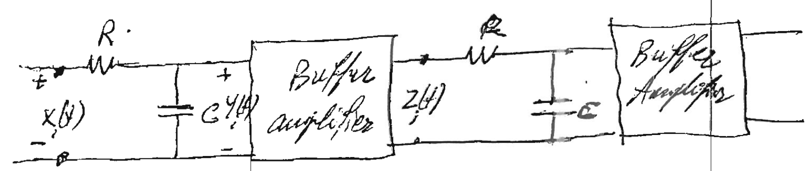
Then $H_1(f) = F.T[h_1(t)]$.

to find $h_1(t)$: if $x(t) = \delta(t)$, then $z(t) = h_1(t)$

That is $h_1(t) = \int_{-\infty}^{t}[\delta(t_1) - \delta(t_1 - T)]dt = ?$

Prob # 2

a)




Note: Buffer Amplifier has unity gain => $z_i(t) = y_i(t)$

Thus the transfer function of the i-th stage is:

$$H_i(f) = \frac{Z_i(f)}{Y_i(f)} = \frac{Y_i(f)}{X_i(f)} = \frac{\frac{1}{j\omega C}}{R + 1/j\omega C} = \frac{1}{1 + j2\pi fRC}$$

EE 443 Hw #4 (Hint) page (2)

prob #2 cont'd)

$$H_i(f) = \frac{1}{1+j2\pi f\tau_0} \quad \text{where } \tau_0 \triangleq RC$$

- Find the overall $H(f)$, that is the transfer function of the cascad of $N$ identical systems.
- Find the amplitud response: that is $|H(f)| = ?$

b) Let $\tau_0^2 \triangleq \frac{\tau^2}{4\pi^2 N}$

Use the definition of $e$ number; that use:

$$\lim_{x\to\infty} \left(1+\frac{\alpha}{x}\right)^{\pm\beta x} \triangleq \exp\left(\pm\frac{\alpha}{x}\beta x\right) = \exp(\alpha\beta)$$

and find $\lim_{N\to\infty} |H(f)| = ?$

prob # 3

a) $g(t) = \text{Sinc}(t)$ , $\hat{g}(t) = ?$

I) Time domain approach

use $\frac{\sin t}{t} \overset{H.T}{\longleftrightarrow} \frac{1-\cos t}{t}$ see prob #4)

Thus $\frac{\sin \pi t}{\pi t} \overset{H.T}{\longleftrightarrow} \frac{1-\cos \pi t}{\pi t}$

use $g_+(t) = g(t) + j\,\hat{g}(t)$ $\Rightarrow$ {Ans $\hat{g}(t) = \text{Sinc}\left(\frac{t}{2}\right) e^{j\frac{\pi}{2}}$ verify.

II) Frequency domain approach:

Find $G(f)$, $G_+(f)$, $g_+(t)$ the find also complex envelope $\tilde{g}(t) = g_+(t)\, e^{-j2\pi f t}$ and

Envelope $a(t) = |\tilde{g}(t)| = |g_+(t)| = ?$

EE 443 | Hw #4 | page 3

4) a) $g(t) = \delta(t)$ , $\hat{g}(t) = ?$

You may use time or frequency domain approach.

b) $g(t) = \frac{\sin t}{t}$  $\hat{g}(t) = ?$ (Ans: $\hat{g}(t) = \frac{1}{t}(1-\cos t)$)

Frequency domain approach: Remember

$\text{rect}(t) \overset{F.T}{\longleftrightarrow} \text{sinc}(f)$

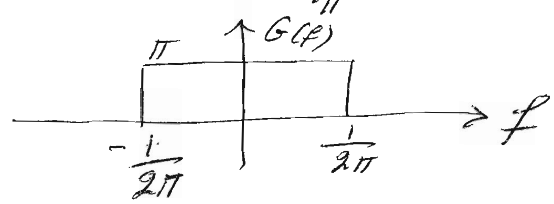
$\text{sinc}(t) \overset{F.T.}{\longleftrightarrow} \text{rect}(-f) = \text{rect}(f)$  Duality

Thus: $\text{sinc}(t) = \frac{\sin \pi t}{\pi t} \longleftrightarrow \text{rect}(f)$

Using time scaling: if $x(t) \overset{F.T}{\longleftrightarrow} X(f)$
then $x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$

in our case $a = \frac{1}{\pi}$ :

$g(t) = \frac{\sin t}{t} \overset{F.T}{\longleftrightarrow} \frac{1}{1/\pi} \text{rec}\left(\frac{f}{1/\pi}\right) = \pi \,\text{rec}\left(\frac{f}{1/\pi}\right) \Longrightarrow$




Now use :

$\hat{G}(f) = -j\,\text{sgn}(f)\,G(f) = -j\pi\,\text{sgn}(f)\,\text{rect}\left(\frac{f}{1/\pi}\right)$

continue !

At some point you may use $\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$

Prob. # 5) Use $R_g(\tau) = F^{-1}T\left[S_g(f)\right]$

where $S_g(f) = \text{rec}\left(\frac{f}{4}\right) + \text{rect}\left(\frac{f}{2}\right)$

### 3.4.2 Problem 1

**Solution** Using transfer function cascading, then the overall transfer function for the system can be written as

$$H(f) = H_1(f)\,H_1(f) = [H_1(f)]^2 \tag{1}$$

Where

$$H_1(f) = \frac{Z(f)}{X(f)}$$

Where

$$\begin{aligned} Z(f) &= \mathcal{F}\left\{\int_{-\infty}^{t} w(\tau)\,d\tau\right\} \\ &= \frac{1}{j2\pi f}W(f) + \frac{W(0)}{2}\delta(f) \end{aligned} \tag{2}$$

Where

$$\begin{aligned} W(f) &= \mathcal{F}\{x(t) - x(t-T)\} \\ &= X(f) - X(f)\,e^{-j2\pi fT} \\ &= X(f)\left[1 - e^{-j2\pi fT}\right] \end{aligned} \tag{3}$$

substitute (3) into (2) we obtain

$$\begin{aligned} Z(f) &= \frac{1}{j2\pi f}X(f)\left[1 - e^{-j2\pi fT}\right] + \frac{X(0)\overbrace{\left[1 - e^{-j2\pi 0T}\right]}^{=0}}{2}\delta(f) \\ &= \frac{1}{j2\pi f}X(f)\left[1 - e^{-j2\pi fT}\right] \end{aligned} \tag{4}$$

Hence

$$\begin{aligned} H_1(f) &= \frac{Z(f)}{X(f)} \\ &= \frac{\frac{1}{j2\pi f}X(f)\left[1 - e^{-j2\pi fT}\right]}{X(f)} \end{aligned}$$

Hence

$$\boxed{H_1(f) = \frac{1}{j2\pi f}\left[1 - e^{-j2\pi fT}\right]}$$

Hence from (1)

$$\begin{aligned} H(f) &= \left(\frac{1}{j2\pi f}\left[1 - e^{-j2\pi fT}\right]\right)^2 \\ &= \frac{1}{-4\pi^2 f^2}\left[1 - e^{-j2\pi fT}\right]^2 \\ &= \frac{-1}{(2\pi f)^2}\left[1 - 2e^{-j2\pi fT} + e^{-j4\pi fT}\right] \end{aligned}$$

Hence

$$\boxed{H\left(f\right)=\frac{1}{\left(2\pi f\right)^{2}}\left[2e^{-j2\pi fT}-e^{-j4\pi fT}-1\right]}$$

## 3.4.3 Problem 2

### 3.4.3.1 Part(a)

Transfer function for each stage is $H_i\left(f\right)=\frac{Y_i(f)}{X_i(f)}=\frac{1}{1+j2\pi fRC}$

Since $RC=\tau_0$, hence

$$H_i\left(f\right)=\frac{1}{1+j2\pi f\tau_0}$$

Then, for $N$ stages, the overall transfer function is

$$H\left(f\right)=H_1\left(f\right)H_2\left(f\right)\cdots H_N\left(f\right)$$

Since they are identical stages, then the transfer function of each stage is the same, and the above becomes

$$H\left(f\right)=\left(\frac{1}{1+j2\pi f\tau_0}\right)^N$$

Hence the amplitude of the response is given by

$$\begin{aligned}\left|H\left(f\right)\right| &= \left(\frac{1}{\left|1+j2\pi f\tau_0\right|}\right)^N\\ &= \left(\frac{1}{\sqrt{1^2+\left(2\pi f\tau_0\right)^2}}\right)^N\\ &= \left(\frac{1}{\left(1+4\pi^2f^2\tau_0^2\right)^{\frac{1}{2}}}\right)^N\\ &= \frac{1}{\left(1+4\pi^2f^2\tau_0^2\right)^{\frac{N}{2}}}\end{aligned}$$

Let $\tau_0^2=\frac{\tau^2}{4\pi^2N}$, the above becomes

$$\left|H\left(f\right)\right|=\frac{1}{\left(1+\frac{f^2\tau^2}{N}\right)^{\frac{N}{2}}} \tag{1}$$

### 3.4.3.2 Part (b)

Let $\alpha=f^2\tau^2,\beta=\frac{1}{2}$, then (1) becomes

$$\left|H\left(f\right)\right|=\frac{1}{\left(1+\frac{\alpha}{N}\right)^{\beta N}}$$

But $\lim_{N\to\infty}\frac{1}{\left(1+\frac{\alpha}{N}\right)^{\beta N}}=e^{\alpha\beta}$,hence

$$\begin{aligned}|H(f)| &= \frac{1}{e^{\frac{f^2\tau^2}{2}}}\\ &= e^{-\frac{f^2\tau^2}{2}}\end{aligned}$$

Which is what we are asked to show.

### 3.4.4 Problem 3

#### 3.4.4.1 Part(a)

(a) $g(t)=\operatorname{sinc}(t)$

$$g_+(t)=g(t)+j\hat{g}(t) \tag{1}$$

Where $\hat{g}(t)$ is Hilbert transform of $g(t)$ defined as $\hat{g}(t)=g(t)\otimes\frac{1}{\pi t}$

$$\begin{aligned}\hat{G}(f) &= -j\ sgn(f)\ G(f)\\ &= -j\ sgn(f)\ rect(f)\end{aligned}$$

Now find the inverse Fourier transform.

I derive the above to answer problem 4 part (b). The answer is the following (please see problem 4 part(b) for the derivation

$$\hat{g}(t)=\frac{1}{\pi t}(1-\cos\pi t)$$

In the above, I used $\operatorname{sinc}(t)\equiv\frac{\sin\pi t}{\pi t}$. If one uses $\operatorname{sinc}(t)\equiv\frac{\sin t}{t}$ then the answer becomes

$$\hat{g}(t)=\frac{1}{t}(1-\cos t) \tag{2}$$

The problem statement seems to want us to use the second definition of $\operatorname{sinc}(t)$, so I will continue the rest of the solution using (1).

Substitute (2) into (1) we obtain

$$\begin{aligned}g_+(t) &= \operatorname{sinc}(t)+j\frac{1}{t}(1-\cos t)\\ &= \frac{\sin(t)}{t}+j\frac{1}{t}\left(1-\frac{e^{jt}+e^{-jt}}{2}\right)\\ &= \frac{1}{t}\frac{e^{jt}-e^{-jt}}{2j}+\frac{1}{t}\left(j+\frac{e^{jt}+e^{-jt}}{2j}\right)\\ &= \frac{1}{t}\frac{e^{jt}-e^{-jt}}{2j}+\frac{j}{t}+\frac{1}{t}\frac{e^{jt}+e^{-jt}}{2j}\\ &= \frac{1}{t}\frac{e^{jt}}{2j}+\frac{j}{t}+\frac{1}{t}\frac{e^{jt}}{2j}\end{aligned}$$

Hence

$$\boxed{g_{+}(t)=\frac{1}{t}\left(j+e^{jt}\right)}$$

#### 3.4.4.2 Part(b)

$$g(t)=[1+k\cos 2\pi f_m t]\cos(2\pi f_c t)$$

$$g_{+}(t)=g(t)+j\hat{g}(t)$$

Where $\hat{g}(t)$ is Hilbert transform of $g(t)$ defined as $\hat{g}(t)=g(t)\otimes\frac{1}{\pi t}$.

$$G_{+}(f)=\begin{cases}2G(f) & f>0\\ G(0) & f=0\\ 0 & f<0\end{cases}$$

But

$$G(f)=\mathcal{F}[1+k\cos 2\pi f_m t]\otimes\mathcal{F}[\cos(2\pi f_c t)] \tag{1}$$

But

$$\mathcal{F}[\cos(2\pi f_c t)]=\frac{1}{2}[\delta(f-f_c)+\delta(f+f_c)]$$

and

$$\mathcal{F}[1+k\cos 2\pi f_m t]=\delta(f)+\frac{k}{2}[\delta(f-f_m)+\delta(f+f_m)]$$

Hence (1) becomes

$$\begin{aligned}G(f)&=\left\{\delta(f)+\frac{k}{2}[\delta(f-f_m)+\delta(f+f_m)]\right\}\otimes\frac{1}{2}[\delta(f-f_c)+\delta(f+f_c)]\\
&=\delta(f)\otimes\frac{1}{2}[\delta(f-f_c)+\delta(f+f_c)]\\
&+\frac{k}{2}[\delta(f-f_m)+\delta(f+f_m)]\otimes\frac{1}{2}[\delta(f-f_c)+\delta(f+f_c)]\end{aligned}$$

$$\begin{aligned}&=\frac{1}{2}\delta(f)\otimes\delta(f-f_c)+\\
&\frac{1}{2}\delta(f)\otimes\delta(f+f_c)+\\
&\frac{k}{4}\delta(f-f_m)\otimes\delta(f-f_c)+\\
&\frac{k}{4}\delta(f-f_m)\otimes\delta(f+f_c)+\\
&\frac{k}{4}\delta(f+f_m)\otimes\delta(f-f_c)+\\
&\frac{k}{4}\delta(f+f_m)\otimes\delta(f+f_c)\end{aligned}$$

Hence

$$
\begin{aligned}
G(f) = &\frac{1}{2}\delta(f+f_c)+ \\
&\frac{1}{2}\delta(f-f_c)+ \\
&\frac{k}{4}\delta(f-f_m+f_c)+ \\
&\frac{k}{4}\delta(f-f_m-f_c)+ \\
&\frac{k}{4}\delta(f+f_m+f_c)+ \\
&\frac{k}{4}\delta(f+f_m-f_c)
\end{aligned}
$$

Hence for $f>0$ ,$G_+(f)=2G(f)$ and we obtain

$$
G_+(f)=\delta(f-f_c)+\frac{k}{2}\left[\delta(f-f_m+f_c)+\delta(f-f_m-f_c)+\delta(f+f_m+f_c)+\delta(f+f_m-f_c)\right]
$$

Then (since carrier frequency $f_c>f_m$), we could simplify the above, by keeping positive frequencies $f$

$$
G_+(f)=\delta(f-f_c)+\frac{k}{2}\left[\delta(f-f_m-f_c)+\delta(f+f_m-f_c)\right]
$$

or

$$
G_+(f)=\delta(f-f_c)+\frac{k}{2}\left[\delta(f-(f_m+f_c))+\delta(f-(f_c-f_m))\right]
$$

Hence

$$
\begin{aligned}
g_+(t) &= e^{j2\pi f_c t}+\frac{k}{2}\left(e^{j2\pi(f_m+f_c)t}+e^{j2\pi(f_c-f_m)t}\right) \\
&= e^{j2\pi f_c t}+\frac{k}{2}\left(e^{j2\pi f_m t}e^{j2\pi f_c t}+e^{j2\pi f_c t}e^{-j2\pi f_m t}\right) \\
&= e^{j2\pi f_c t}\left[1+\frac{k}{2}\left(e^{j2\pi f_m t}+e^{-j2\pi f_m t}\right)\right] \\
&= e^{j2\pi f_c t}\left[1+\frac{k}{2}\left(2\cos(2\pi f_m t)\right)\right] \\
&= e^{j2\pi f_c t}\left[1+k\cos(2\pi f_m t)\right]
\end{aligned}
$$

## 3.4.5 Problem 4

### 3.4.5.1 Part(a)

$g(t) = \delta(t)$

$$\begin{aligned}\hat{g}(t) &= g(t) \otimes \frac{1}{\pi t} \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \delta(\tau)\frac{1}{t-\tau}d\tau \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \delta(\tau)\frac{1}{t}d\tau \\ &= \frac{1}{\pi t}\int_{-\infty}^{\infty} \delta(\tau)\, d\tau \\ &= \frac{1}{\pi t}\end{aligned}$$

### 3.4.5.2 Part(b)

And Since $sgn(f) = -1$ for $f < 0$ and $sgn(f) = 1$ for $f > 0$ then

$$\hat{G}(f) = -j\left[-rect\left(\frac{f+\frac{1}{4}}{\frac{1}{2}}\right) + rect\left(\frac{f-\frac{1}{4}}{\frac{1}{2}}\right)\right]$$

Hence

$$\hat{g}(t) = jF^{-1}\left[rect\left(\frac{f+\frac{1}{4}}{\frac{1}{2}}\right) - rect\left(\frac{f-\frac{1}{4}}{\frac{1}{2}}\right)\right] \tag{1}$$

But $F^{-1}\left(rect\left(\frac{f+\frac{1}{4}}{\frac{1}{2}}\right)\right) = \frac{1}{2}\operatorname{sinc}\left(\frac{1}{2}t\right)e^{-j2\pi\frac{1}{4}t}$ and $F^{-1}\left(rect\left(\frac{f-\frac{1}{4}}{\frac{1}{2}}\right)\right) = \frac{1}{2}\operatorname{sinc}\left(\frac{1}{2}t\right)e^{+j2\pi\frac{1}{4}t}$, hence (1) becomes

$$\begin{aligned}\hat{g}(t) &= j\left[\frac{1}{2}\operatorname{sinc}\left(\frac{1}{2}t\right)e^{-j2\pi\frac{1}{4}t} - \frac{1}{2}\operatorname{sinc}\left(\frac{1}{2}t\right)e^{+j2\pi\frac{1}{4}t}\right] \\ &= \frac{1}{2}\operatorname{sinc}\left(\frac{1}{2}t\right)\left[j\left(e^{-j2\pi\frac{1}{4}t} - e^{j2\pi\frac{1}{4}t}\right)\right] \\ &= \frac{1}{2}\operatorname{sinc}\left(\frac{1}{2}t\right)\left[\frac{e^{-j\frac{\pi}{2}t} - e^{j\frac{\pi}{2}t}}{-j}\right] \\ &= \operatorname{sinc}\left(\frac{1}{2}t\right)\left[\frac{e^{j\frac{\pi}{2}t} - e^{-j\frac{\pi}{2}t}}{2j}\right] \\ &= \operatorname{sinc}\left(\frac{1}{2}t\right)\left[\sin\frac{\pi}{2}t\right]\end{aligned}$$

But $\operatorname{sinc}\left(\frac{1}{2}t\right) = \frac{\sin\frac{\pi t}{2}}{\frac{\pi t}{2}}$ hence

$$\begin{aligned}
\hat{g}(t) &= \frac{\sin\frac{\pi t}{2}}{\frac{\pi t}{2}} \sin\frac{\pi}{2}t \\
&= \frac{2}{\pi t}\sin^2\frac{\pi}{2}t \\
&= \frac{2}{\pi t}\left(\frac{1}{2} - \frac{1}{2}\cos\pi t\right) \\
&= \frac{1}{\pi t}\left(1 - \cos\pi t\right)
\end{aligned}$$

### 3.4.6 problem 5

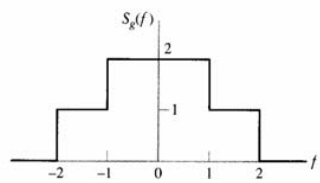



**FIGURE 2.47**
Problem 2.44

**2.45** Consider the square wave $g(t)$ shown in Fig. 2.48. Find the power spectral density, average power, and autocorrelation function of this square wave. Does the wave have dc power? Explain your answer.

Figure 3.11: the Problem statement

$$S_g(f) = rect\left(\frac{f}{4}\right) + rect\left(\frac{f}{2}\right)$$

$$R_g(\tau) = F^{-1}\left(S_g(f)\right)$$

Hence

$$\begin{aligned}
R_g(\tau) &= F^{-1}\left(rect\left(\frac{f}{4}\right) + rect\left(\frac{f}{2}\right)\right) \\
&= F^{-1}\left[rect\left(\frac{f}{4}\right)\right] + F^{-1}\left[rect\left(\frac{f}{2}\right)\right] \\
&= 4\operatorname{sinc}(4t) + 2\operatorname{sinc}(2t)
\end{aligned}$$

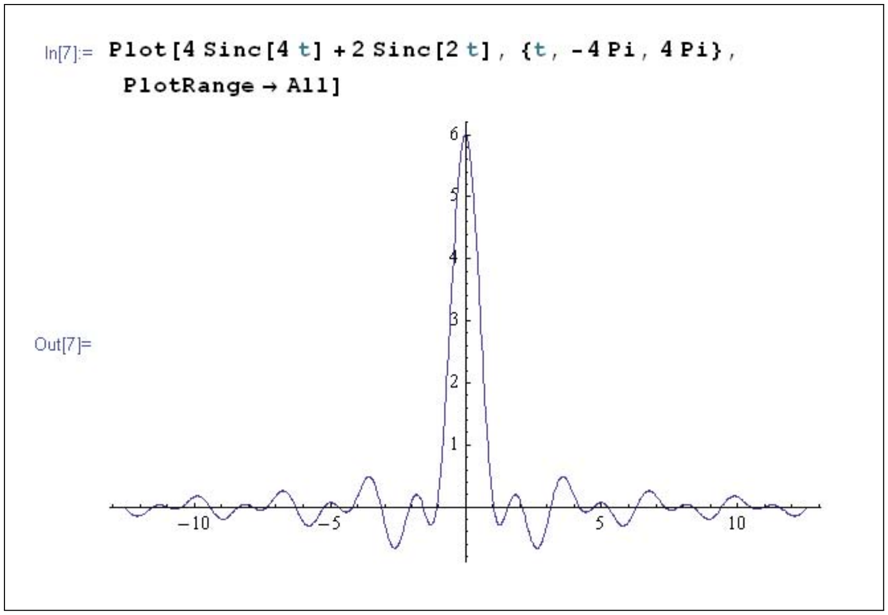
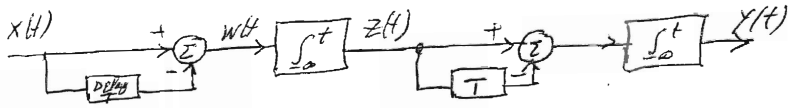



Figure 3.12: Plot for problem 5

### 3.4.7 Key solution

EE 443 HW #4 Key page




**Problem**

The first integrator input is equal to x(t)-x(t-T). The Fourier transform of this input signal is [1-exp(-j2πfT)]X(f). The value of this transform is zero at f=0. It follows therefore that the Fourier transform of the first integrator output is equal to

That is: $w(t) = x(t) - x(t-T) \Rightarrow W(f) = X(f)[1-e^{-j2\pi f}$

$z(t) = \int_{-\infty}^{t} w(t_1)dt_1 \Rightarrow Z(f) = \frac{W(f)}{j2\pi f} + \frac{1}{2}W(0)\delta(f)$

Since $W(0)=0 \Rightarrow Z(f) = \frac{W(f)}{j2\pi f} \rightarrow (*)$

$$(*)\quad Z(f) = \frac{1}{j2\pi f}[1-\exp(-j2\pi fT)]X(f)$$

The transfer function of the first stage of the system of Fig. P2.5 is therefore equal to $H_1(f) = \frac{Z(f)}{X(f)}$

$$\Rightarrow H_1(f) = \frac{1}{j2\pi f}[1-\exp(-j2\pi fT)] = \frac{e^{-j\pi fT}}{j2\pi f}\left[e^{j\pi fT} - e^{-j\pi fT}\right] = \frac{e^{-j\pi fT}}{\pi f}\sin(\pi fT) = Te^{-j\pi fT}\operatorname{sinc}($$

The second stage of the system is identical to the first stage. The overall transfer function of the system is therefore : $H(f) = H_1(f)\times H_1(f)$

$$H(f) = \frac{1}{(j2\pi f)^2}[1-\exp(-j2\pi fT)]^2 = \frac{1 + e^{-j4\pi fT} - 2e^{-j2\pi fT}}{(j2\pi f)^2} = \frac{e^{-j2\pi fT}\left[e^{+j2\pi fT} + e^{-j2\pi fT} - 2\right]}{(j2\pi fT)^2}$$

$$= \exp(-j2\pi fT)\left[\frac{\exp(j\pi fT)-\exp(-j\pi fT)}{j2\pi f}\right]^2$$

$$= \exp(-j2\pi fT)\left[\frac{\sin(\pi fT)}{\pi f}\right]^2$$

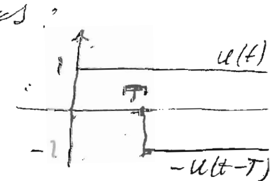
$$= T^2\operatorname{sinc}^2(fT)\exp(-j2\pi fT)$$

**2° method**

The impulse response of the first half:

if $x(t) = \delta(t)$, then $z(t) = h_1(t)$, Thus:

$$h_1(t) = \int_{-\infty}^{t}[\delta(t_1) - \delta(t_1 - T)]dt_1 = u(t) - u(t-T)$$




$$\Rightarrow h_1(t) = \operatorname{rect}\left(\frac{t-T/2}{T}\right)$$

$$\Rightarrow H_1(f) = F\cdot T[h_1(t)] = T\operatorname{Sinc}(fT)\,e^{-j\pi fT}$$

$$H(f) = H_1^2(f) = T^2\operatorname{Sinc}^2(fT)\,e^{-j2\pi fT}$$

Problem 2.

(a) The transfer function of the ith stage of the system of Fig. P2.6 is

$$H_i(f) = \frac{1}{1+j2\pi f RC}$$

$$= \frac{1}{1+j2\pi f \tau_0}$$

where it is assumed that the buffer amplifier has a constant gain of one. The overall transfer function of the system is therefore

$$H(f) = \prod_{i=1}^{N} H_i(f)$$

$$= \frac{1}{(1+j2\pi f \tau_0)^N}$$

The corresponding amplitude response is

$$|H(f)| = \frac{1}{[1+(2\pi f \tau_0)^2]^{N/2}}$$

(b) Let

$$\tau_0^2 = \frac{T^2}{4\pi^2 N}$$

Then, we may rewrite the expression for the amplitude response as

$$|H(f)| = \left[1+ \frac{1}{N}(fT)^2\right]^{-N/2}$$

In the limit, as N approaches infinity, we have

$$|H(f)| = \lim_{N\to\infty} [1+ \frac{1}{N}(fT)^2]^{-N/2}$$

$$= \exp[-\frac{N}{2} \cdot \frac{1}{N}(fT)^2]$$

$$= \exp(-\frac{f^2T^2}{2})$$

Problem 3

(a) $g(t) = \text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$

We note that

$$\hat{g}(t) = \frac{1-\cos(\pi t)}{\pi t}$$

Therefore,

$$g_+(t) = g(t) + j\hat{g}(t)$$

$$= \frac{\sin(\pi t)}{\pi t} + j\,\frac{1-\cos(\pi t)}{\pi t}$$

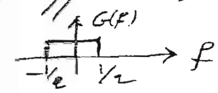
$$= \frac{j}{\pi t}[1 - \cos(\pi t) - j\sin(\pi t)]$$

$$= \frac{j}{\pi t}[1 - \exp(j\pi t)] = \frac{1}{j\pi t}\left[e^{j\pi t} - 1\right] = \frac{1}{j\pi t}\left[\frac{e^{j\pi t} - 1}{e^{j\pi t/2}}\right]e^{j\pi t/2} = \frac{2}{\pi t}\left(\sin \pi t/2\right)\cdot e^{j\ldots} = \text{Sinc}(t/2)\, e^{j\ldots}$$

II° method:

using freq. domain approach:

$G(f) = \text{rect}(f)$




$G_+(f) = 2G(f) = 2\,\text{rect}\left(\frac{f - 1/4}{1/2}\right)$ for $f$...

$g_+(t) = 2\cdot\frac{1}{2}\,\text{Sinc}\left(\frac{t}{2}\right) e^{j2\pi\frac{1}{4}t}$

$\Rightarrow g_+(t) = \text{Sinc}(t/2)\; e^{j2\pi\frac{1}{4}t}$ $f_c =$

$\tilde{g}(t) = g_+(t)\, e^{-j2\pi f_c t} = \text{Sinc}(t/2)$

$a(t) = |\tilde{g}(t)| = |\text{Sinc}(t/2)|$

(b) $g(t) = [1+k\cos(2\pi f_m t)]\cos(2\pi f_c t)$

$$= \cos(2\pi f_c t) + \frac{k}{2}\cos[2\pi(f_c+f_m)t] + \frac{k}{2}\cos[2\pi(f_c-f_m)t]$$

Since the Hilbert transform of $\cos(2\pi ft)$ is equal to $\sin(2\pi ft)$, it follows that

$$\hat{g}(t) = \sin(2\pi f_c t) + \frac{k}{2}\sin[2\pi(f_c+f_m)t] + \frac{k}{2}\sin[2\pi(f_c-f_m)t]$$

where it is assumed that $f_c > f_m$. Therefore,

$$g_+(t) = \exp(j2\pi f_c t) + \frac{k}{2}\exp[j2\pi(f_c+f_m)t] + \frac{k}{2}\exp[j2\pi(f_c-f_m)t]$$

$$= [1 + \frac{k}{2}\exp(j2\pi f_m t) + \frac{k}{2}\exp(-j2\pi f_m t))]\exp(j2\pi f_c t)$$

$$= [1 + k\cos(2\pi f_m t)]\exp(j2\pi f_c t)$$

HW #4 page

**Problem 4**

Prob #4 part b)
Time domain approach:

(b) $g(t) = \frac{\sin t}{t}$

The Hilbert transform of sin t/t is

$$\hat{g}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g(\tau)}{t-\tau}\, d\tau$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin\tau}{\tau(t-\tau)}\, d\tau$$

$$= \frac{1}{\pi t}\int_{-\infty}^{\infty} \left(\frac{1}{\tau} + \frac{1}{t-\tau}\right)\sin\tau\, d\tau$$

$$= \frac{1}{\pi t}\int_{-\infty}^{\infty} \frac{\sin\tau}{\tau}\, d\tau + \frac{1}{\pi t}\int_{-\infty}^{\infty} \frac{\sin\tau}{t-\tau}\, d\tau$$

We note that

$$\int_{-\infty}^{\infty} \operatorname{sinc}(t)\, dt = 1$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{\sin\tau}{\tau}\, d\tau = \pi$$

$$\int_{-\infty}^{\infty} \frac{\sin\tau}{t-\tau}\, d\tau = \int_{-\infty}^{\infty} \frac{\sin(t-\tau)}{\tau}\, d\tau$$

$$= \sin t \underbrace{\int_{-\infty}^{\infty} \frac{\cos\tau}{\tau}\, d\tau}_{0 \text{ it is odd function}} - \cos t\int_{-\infty}^{\infty} \frac{\sin\tau}{\tau}\, d\tau$$

$$= -\pi \cos t$$

We thus obtain

$$\hat{g}(t) = \frac{1}{t}(1-\cos t)$$

Thus $\hat{g}(t) = \frac{1}{t}(1-\cos t)$

Prob. #4 part b

II) Freq. domain approach

Remember: $\operatorname{rect}(t) \xleftrightarrow{F.T} \operatorname{Sinc}(f)$

using duality:

$\operatorname{Sinc}(t) \xleftrightarrow{F.T} \operatorname{rect}(-f) = \operatorname{rect}(f)$

Thus:

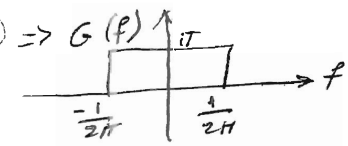
$\operatorname{Sinc}(t) = \frac{\sin \pi t}{\pi t} \longleftrightarrow \operatorname{rect}(f)$

$g(t) = \frac{\sin t}{t} = \frac{\sin(\pi t \cdot \frac{1}{\pi})}{(\pi t \cdot \frac{1}{\pi})} \longleftrightarrow \frac{1}{(1/\pi)} \operatorname{rect}(\ldots$

where time scaling is used. That is if $g(t) \longleftrightarrow G(f)$, then $g(at) \longleftrightarrow \frac{1}{a} G(f/a)$. Therefore in our case:

$g(t) = \frac{\sin t}{t} \xleftrightarrow{F.T} \pi \operatorname{rect}\left(\frac{f}{1/\pi}\right)$

$\Rightarrow G(f)$




$\hat{G}(f) = -j\operatorname{sgn}(f)\, G(f) =$

$= -j\pi \operatorname{sgn}(f)\left[\operatorname{rect}\left(\frac{f-\frac{1}{4\pi}}{1/2\pi}\right) + \operatorname{rect}\left(\frac{f+\frac{1}{4\pi}}{1/2\pi}\right)\right]$

$= -j\pi\left[\operatorname{rect}\left(\frac{f-\frac{1}{4\pi}}{1/2\pi}\right) - \operatorname{rect}\left(\frac{f+\frac{1}{4\pi}}{1/2\pi}\right)\right]$

$\Rightarrow \hat{g}(t) = -j\pi\left[\frac{1}{2\pi}\operatorname{Sinc}\left(\frac{t}{2\pi}\right)\right]\left[e^{jt/2} - e^{-j\ldots}\right]$

$\hat{g}(t) = \operatorname{Sinc}\left(\frac{t}{2\pi}\right)\sin(t/2)$

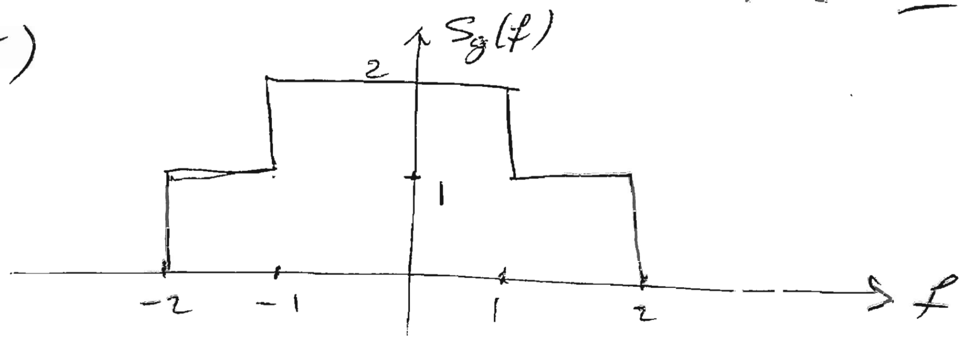
$= \frac{2}{t}\sin^2(t/2) = \frac{1}{t}(1-\cos t)$

a) $g(t) = \delta(t)$

$$\hat{g}(t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\delta(\tau)}{t-\tau}\, d\tau = \frac{1}{\pi t}\int_{-\infty} \delta(\tau)\, d\tau = \frac{1}{\pi t}$$

HW #4 page 5

prob #5)




$$S_g(f) = rect\left(\frac{f}{4}\right) + rect\left(\frac{f}{2}\right)$$

$$R_g(\tau) = F^{-1}\left[S_g(f)\right] = 4\, Sinc(4\tau) + 2\, Sinc(2\tau)$$

$$R_g(0) = P_{av} = 6 \text{ watts}$$

That is

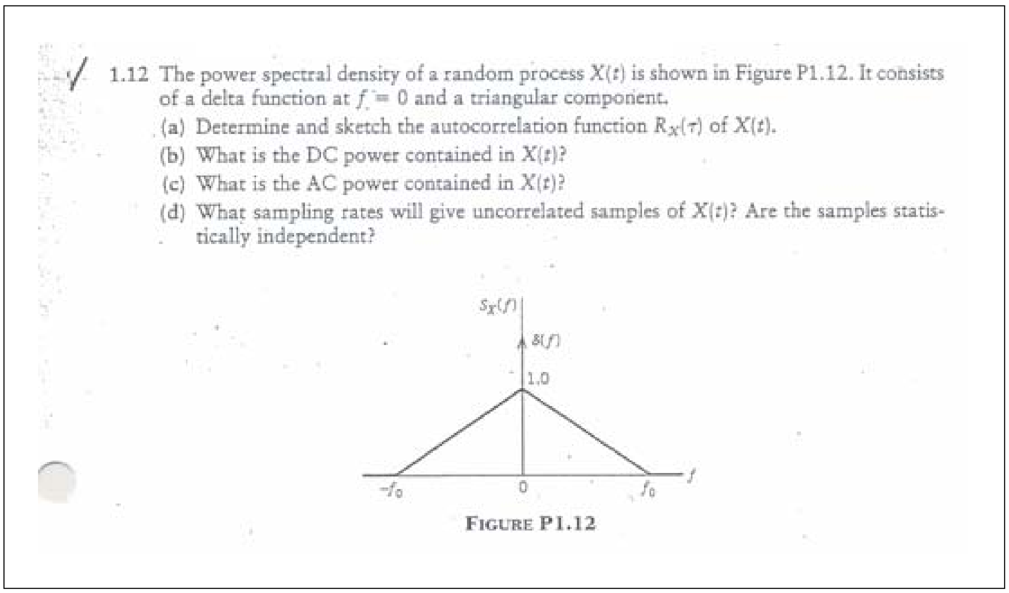
$$R_g(0) = \int_{-\infty}^{+\infty} S_g(f)\, df = 6\ W \triangleq P_{av}$$

# 3.5 HW 5

## Local contents



## 3.5.1 Problem 1



1.12 The power spectral density of a random process $X(t)$ is shown in Figure P1.12. It consists of a delta function at $f = 0$ and a triangular component.

(a) Determine and sketch the autocorrelation function $R_X(\tau)$ of $X(t)$.

(b) What is the DC power contained in $X(t)$?

(c) What is the AC power contained in $X(t)$?

(d) What sampling rates will give uncorrelated samples of $X(t)$? Are the samples statistically independent?



FIGURE P1.12

Figure 3.13: the Problem statement

#### 3.5.1.1 Part(a)

Assuming stationary process,

$$R_x(\tau) \Leftrightarrow S_x(f)$$

But $S_x(f) = \delta(f) + tri\left(\frac{f}{2f_0}\right)$, hence

$$R_x(\tau) = F^{-1}\left(\delta(f) + tri\left(\frac{f}{2f_0}\right)\right)$$
$$= \int_{-\infty}^{\infty} \left[\delta(f) + tri\left(\frac{f}{2f_0}\right)\right] e^{j2\pi f\tau} df$$

But $F^{-1}\left(tri\left(\frac{f}{2f_0}\right)\right) = f_0 \frac{\sin^2(f_0\pi\tau)}{f_0^2\pi^2\tau^2}$, and $F^{-1}(\delta(f)) = 1$, hence the above becomes

Hence

$$R_x(\tau) = \overbrace{1}^{\text{dc part}} + \overbrace{f_0 \operatorname{sinc}^2(f_0\tau)}^{\text{AC part}}$$

#### 3.5.1.2 Part(b)

$$P_x(0) = 1 + f_0$$

Hence DC power in $X(t)$ is given 1 watt.

#### 3.5.1.3 Part(c)

The AC power is $f_0$ watt.

#### 3.5.1.4 Part(d)

Since $R_x(\tau) = 1 + f_0 \operatorname{sinc}^2(f_0\tau)$, we need to make this zero. But this has no real root as solution (assuming $f_0 \geq 0$)

To obtain a solution, I will only consider the AC part.

Hence we need to solve for $\tau$ in

$$R_x(\tau) = f_0 \operatorname{sinc}^2(f_0\tau) = 0$$

i.e. the AC part only.

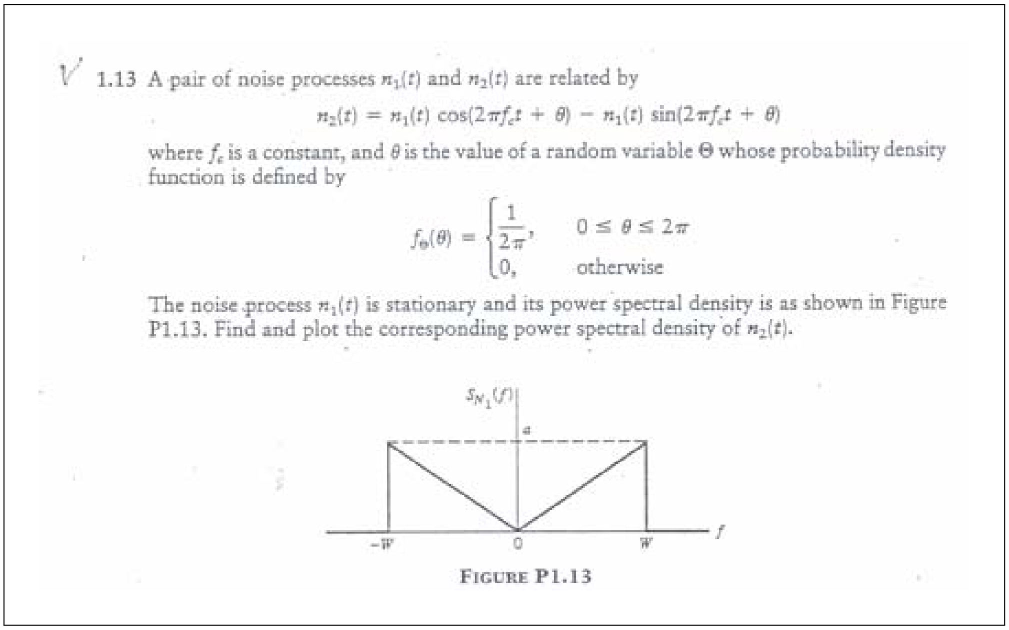
This is zero when $\operatorname{sinc}^2(f_0\tau) = 0$ or when $\sin(\pi f_0\tau) = 0$ or when

$\pi f_0\tau = k\pi$, $k = \pm 1, \pm 2, \cdots$.

Hence when

$$\tau = \pm\frac{1}{f_0}, \pm\frac{2}{f_0}, \cdots$$

### 3.5.2 Problem 2



1.13 A pair of noise processes $n_1(t)$ and $n_2(t)$ are related by

$$n_2(t) = n_1(t)\cos(2\pi f_c t + \theta) - n_1(t)\sin(2\pi f_c t + \theta)$$

where $f_c$ is a constant, and $\theta$ is the value of a random variable $\Theta$ whose probability density function is defined by

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \le \theta \le 2\pi \\ 0, & \text{otherwise} \end{cases}$$

The noise process $n_1(t)$ is stationary and its power spectral density is as shown in Figure P1.13. Find and plot the corresponding power spectral density of $n_2(t)$.



FIGURE P1.13

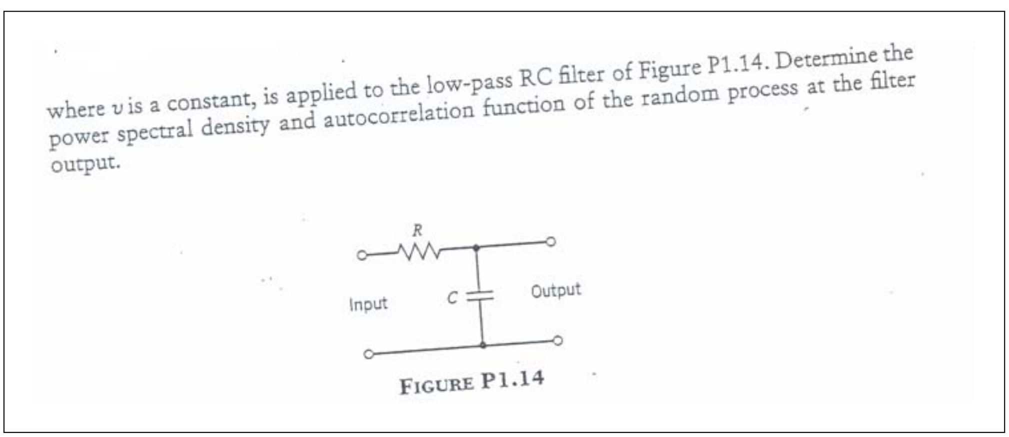
Figure 3.14: the Problem statement

(see graded HW for solution)

### 3.5.3 Problem 3

A random telegraph signal $X(t)$ charaterized by the autocorrelation function

$$R_X(\tau) - e^{-2\nu|tau|}$$



where $\upsilon$ is a constant, is applied to the low-pass RC filter of Figure P1.14. Determine the power spectral density and autocorrelation function of the random process at the filter output.



FIGURE P1.14

Figure 3.15: the Problem statement

Let $S_y(f)$ be the psd of the output, then

$$S_y(f) = S_x(f)\,|H(f)|^2$$

But

$$\begin{aligned}
S_x(f) &= \mathcal{F}(R_x(\tau)) \\
&= \int_{-\infty}^{0} e^{2v\tau} e^{-j2\pi f\tau} d\tau + \int_{0}^{\infty} e^{-2v\tau} e^{-j2\pi f\tau} d\tau \\
&= \int_{-\infty}^{0} e^{\tau(2v-j2\pi f)} d\tau + \int_{0}^{\infty} e^{\tau(-2v-j2\pi f)} d\tau \\
&= \frac{\left[e^{\tau(2v-j2\pi f)}\right]_{-\infty}^{0}}{2v-j2\pi f} + \frac{\left[e^{\tau(-2v-j2\pi f)}\right]_{0}^{\infty}}{-2v-j2\pi f} \\
&= \frac{1}{2v-j2\pi f} + \frac{-1}{-2v-j2\pi f} \\
&= \frac{1}{2v-j2\pi f} + \frac{1}{2v+j2\pi f} \\
&= \frac{4v}{4v^2+4\pi^2 f^2}
\end{aligned}$$

Now we need to find $H(f)$. Using voltage divider $H(f) = \frac{Y(f)}{X(f)} = \frac{\frac{1}{j2\pi fC}}{R+\frac{1}{j2\pi fC}}$

hence

$$H(f) = \frac{1}{j2\pi fRC+1}$$

Hence

$$|H(f)| = \frac{1}{\sqrt{1+(2\pi fRC)^2}}$$

Then

$$\begin{aligned}
S_y(f) &= S_x(f)\left|H(f)\right|^2 \\
&= \left(\frac{4v}{4v^2+4\pi^2 f^2}\right)\left(\frac{1}{1+(2\pi f RC)^2}\right) \\
&= \frac{4v}{(4v^2+4\pi^2 f^2)(1+4\pi^2 f^2 R^2 C^2)} \\
&= \frac{4v}{4v^2+4v^2(2\pi f RC)^2+4\pi^2 f^2+4\pi^2 f^2(2\pi f RC)^2} \\
&= \frac{4v}{4v^2+16v^2\pi^2 f^2 R^2 C^2+4\pi^2 f^2+16\pi^2 f^2 \pi^2 f^2 R^2 C^2} \\
&= \frac{v}{v^2+4v^2\pi^2 f^2 R^2 C^2+\pi^2 f^2+4\pi^4 f^4 R^2 C^2}
\end{aligned}$$

Now, $R_y(\tau)$ is the inverse Fourier transform of the above.

### 3.5.4 Problem 4

1.15 A *running integrator* is defined by

$$y(t) = \int_{t-T}^{t} x(\tau)\, d\tau$$

where $x(t)$ is the input, $y(t)$ is the output, and $T$ is the integration period. Both $x(t)$ and $y(t)$ are sample functions of stationary processes $X(t)$ and $Y(t)$, respectively. Show that the power spectral density of the integrator output is related to that of the integrator input as

$$S_Y(f) = T^2 \operatorname{sinc}^2(fT) S_X(f)$$

Figure 3.16: the Problem statement

(see graded HW for solution)

### 3.5.5 Key solution

Missing Solutions for HW#5

page 1

EE 443 HW#5

1.15)

$$y(t) = \int_{t-T}^{t} x(\tau_1)\, d\tau_1 \qquad (1)$$

Method # 1

When $x(t) = \delta(t) \Rightarrow y(t) = h(t)$, Thus

$$h(t) = \int_{t-T}^{t} \delta(t)\, dt = u(t) - u(t-T) = rect\left(\frac{t-T/2}{T}\right)$$

Thus $H(f) = F.T[h(t)] = T\, Sinc(fT)\, e^{-j\pi f T}$

Method # 2

Differentiate eq (1):

$$\frac{dy(t)}{dt} = x(t) - x(t-T) \qquad (2)$$

Take F.T of eq (2)

$$Y(f) \cdot j2\pi f = X(f) - X(f)\, e^{-j2\pi f T}$$

$$Y(f) = \frac{X(f)}{j2\pi f}\left[1 - e^{-j2\pi f T}\right] = \frac{X(f)}{j2\pi f} \cdot e^{-j\pi f T}\left[e^{+j\pi f T} - e^{-j\pi f T}\right]$$

$$= X(f) \cdot T\, e^{-j\pi f T}\, Sinc(fT)$$

$$\Rightarrow Y(f) = X(f) \cdot T\, e^{-j\pi f T}\, Sinc(fT)$$

$$\frac{Y(f)}{X(f)} = T\, e^{-j\pi f T} \ldots$$

EE 443 HW page 1

Problem 1.17

The autocorrelation function of X(t) is

$$R_X(\tau) = E[X(t+\tau)\,X(t)]$$
$$= A^2\,E[\cos(2\pi Ft + 2\pi F\tau - \Theta)\cos(2\pi Ft - \Theta)]$$
$$= \frac{A^2}{2}\,E[\cos(4\pi Ft + 2\pi F\tau - 2\Theta) + \cos(2\pi F\tau)]$$

Averaging over Θ, and noting that Θ is uniformly distributed over 2π radians, we get

$$R_X(\tau) = \frac{A^2}{2}\,E[\cos(2\pi F\tau)]$$
$$= \frac{A^2}{2}\int_{-\infty}^{\infty} f_F(f)\cos(2\pi f\tau)\,df$$

Next, we note that $R_X(\tau)$ is related to the power spectral density by

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)\cos(2\pi f\tau)\,df$$

Therefore, comparing Eqs. (1) and (2), we deduce that the power spectral density of X(t) is

$$S_X(f) = \frac{A^2}{2} f_F(f)$$

When the frequency assumes a constant value, $f_c$ (say), we have

$$f_F(f) = \frac{1}{2}\delta(f-f_c) + \frac{1}{2}\delta(f+f_c)$$

E12

Thus: $S_X(f) = \frac{A^2}{4}\{\delta(f-f_c) + \delta(f+f_c)\}$

19

EE 443 HW # 6
Key

8.35

$$S_X(f) = Tri(f) = \begin{cases} 1-|f|, & |f| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$R_X(\tau) = F^{-1}T\left[S_X(f)\right] = \text{Sinc}^2(\tau)$$

since $Tri(t) \overset{F.T}{\longleftrightarrow} \text{Sinc}^2(f)$

8.32)

$$R_X(\tau) = \begin{cases} \sigma^2(1-|\tau|) & |\tau| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

using $Tri(t) \longleftrightarrow \text{Sinc}^2(f)$

$R_X(\tau) = \sigma^2 \, Tri(\tau)$ thus;

$$S_X(f) = F.T\left[R_X(\tau)\right] = \sigma^2 \text{Sinc}^2(f)$$

EE 443 Chapt. 8 HW #6 3

✓ Problem 1.12

(a) The power spectral density consists of two components:

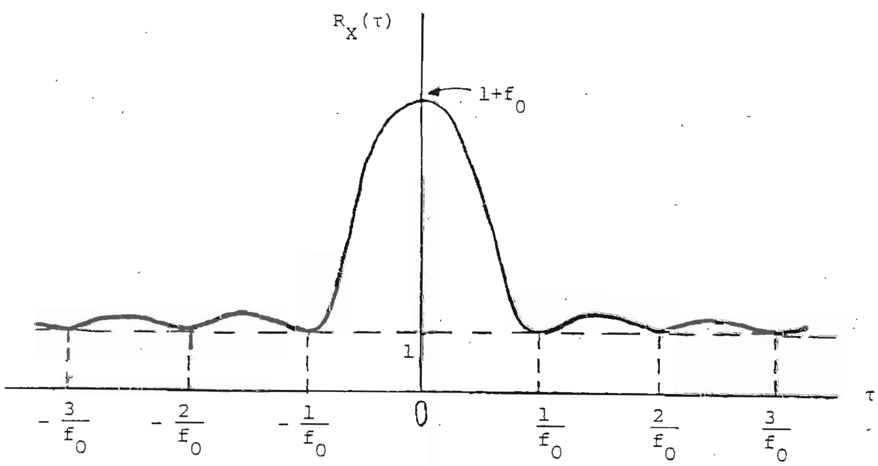
(1) A delta function $\delta(t)$ at the origin, whose inverse Fourier transform is one.

(2) A triangular component of unit amplitude and width $2f_0$, centered at the origin; the inverse Fourier transform of this component is $f_0 \operatorname{sinc}^2(f_0\tau)$.

Therefore, the autocorrelation function of X(t) is

$$R_X(\tau) = 1 + f_0 \operatorname{sinc}^2(f_0\tau)$$

which is sketched below:




$= \cos[2\pi(t_1-t_2)]$

(b) Since $R_X(\tau)$ contains a constant component of amplitude 1, it follows that the dc power contained in X(t) is 1.

(c) The mean-square value of X(t) is given by

$$E[X^2(t)] = R_X(0)$$
$$= 1 + f_0$$

The ac power contained in X(f) is therefore equal to $f_0$.

(d) If the sampling rate is $f_0/n$, where n is an integer, the samples are uncorrelated. They are not, however, statistically independent. They would be statistically independent if X(t) were a Gaussian process.

## 3.6 HW 6

### Local contents



HW and key are missing.

### 3.6.1 Questions



8.35 Consider a wide-sense stationary process $X(t)$ having the power spectral density $S_X(f)$ shown in Fig. 8.26. Find the autocorrelation function $R_X(\tau)$ of the process $X(t)$.

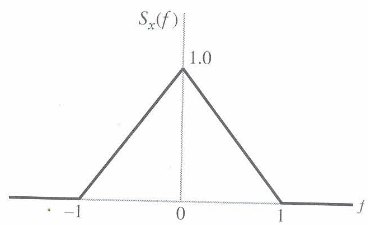



**FIGURE 8.26** Problem 8.35.

8.36 The power spectral density of a random process $X(t)$ is shown in Fig. 8.27.
(a) Determine and sketch the autocorrelation function $R_X(\tau)$ of $X(t)$.
(b) What is the dc power contained in $X(t)$?
(c) What is the ac power contained in $X(t)$?
(d) What sampling rates will give uncorrelated samples of $X(t)$? Are the samples statistically independent?

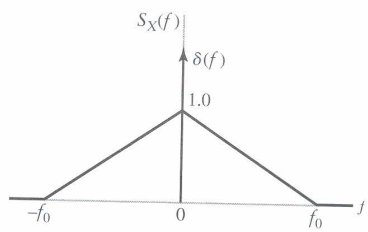



**FIGURE 8.27** Problem 8.36.

8.37 Consider the two linear filters shown in cascade as in Fig. 8.28. Let $X(t)$ be a stationary process with autocorrelation function $R_X(\tau)$. The random process appearing at the first filter output is $V(t)$ and that at the second filter output is $Y(t)$.
(a) Find the autocorrelation function of $V(t)$.
(b) Find the autocorrelation function of $Y(t)$.

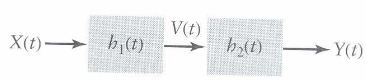



**FIGURE 8.28** Problem 8.37.

8.38 The power spectral density of a narrowband random process $X(t)$ is as shown in Fig. 8.29. Find the power spectral densities of the in-phase and quadrature components of $X(t)$, assuming $f_c = 5$ Hz.

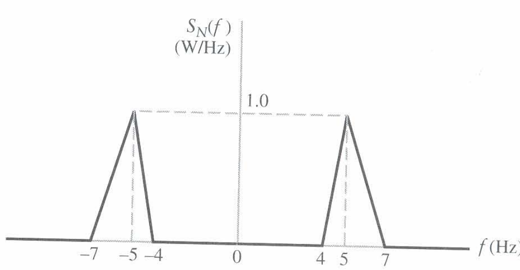



**FIGURE 8.29** Problem 8.38.



Figure 3.17: the Problem statement

## 3.7 HW 7

### Local contents

### 3.7.1 Questions

Book Coash ?
HW question. Used for HW's 7, 8, 9, 10



$$(f_c)_{SSB} - f_1 = 7090 \text{ kHz} - 2.225 \text{ kHz} = 7087.775 \text{ kHz}$$

a space frequency (binary 0) of

$$(f_c)_{SSB} - f_2 = 7090 - 2.025 = 7087.975 \text{ kHz}$$

and a carrier frequency of

$$(f_c)_{FSK} = (f_c)_{SSB} - (f_c)_{Bell\ 103} = 7090 - 2.125 = 7087.875 \text{ kHz}$$

Consequently, the SSB transceiver would produce a FSK digital signal with a carrier frequency of 7087.875 kHz.

For the case of alternating data, the spectrum of this FSK signal is given by (5-85) and (5-86), where $f_c$ = 7087.875 kHz. The resulting spectral plot would be like that of Fig. 5-26a, where the spectrum is translated from $f_c$ = 1170 Hz to $f_c$ = 7087.875 kHz. It is also realized that this spectrum appears on the lower sideband of the SSB carrier frequency $(f_c)_{SSB}$ = 7090 kHz. If a DSB-SC transmitter had been used (instead of a LSSB transmitter), the spectrum would be replicated on the upper sideband as well as on the lower sideband, and two redundant FSK signals would be emitted.

For the case of random data, the PSD for the complex envelope is given by (5-90) and shown in Fig. 5-25 for the modulation index of $h$ = 0.7. Using (5-2b), the PSD for the FSK signal is the translation of the PSD for the complex envelope to the carrier frequency of 7087.875 kHz.

**5-1** An AM broadcast transmitter is tested by feeding the RF output into a 50-Ω (dummy) load. Tone modulation is applied. The carrier frequency is 850 kHz and the FCC licensed power output is 5000 W. The sinusoidal tone of 1000 Hz is set for 90% modulation.
**(a)** Evaluate the FCC power in dBk (dB above 1 kW) units.
**(b)** Write an equation for the voltage that appears across the 50-Ω load, giving numerical values for all constants.
**(c)** Sketch the spectrum of this voltage as it would appear on a calibrated spectrum analyzer.
**(d)** What is the average power that is being dissipated in the dummy load?
**(e)** What is the peak envelope power?

**5-2** An AM transmitter is modulated with an audio testing signal given by $m(t) = 0.2 \sin \omega_1 t + 0.5 \cos \omega_2 t$, where $f_1 = 500$ Hz, $f_2 = 500\sqrt{2}$ Hz, and $A_c = 100$. Assume that the AM signal is fed into a 50-Ω load.
**(a)** Sketch the AM waveform.
**(b)** What is the modulation percentage?
**(c)** Evaluate and sketch the spectrum of the AM waveform.

**5-3** For the AM signal given in Prob. 5-2:
**(a)** Evaluate the average power of the AM signal.
**(b)** Evaluate the PEP of the AM signal.

is this Normalized power or actual ...

**5-4** Assume that an AM transmitter is modulated with a video testing signal given by $m(t) = -0.2 + 0.6 \sin \omega_1 t$ where $f_1$ = 3.57 MHz. Let $A_c$ = 100.
**(a)** Sketch the AM waveform.
**(b)** What is the percentage of positive and negative modulation?
**(c)** Evaluate and sketch the spectrum of the AM waveform about $f_c$.

**5-5** A 50,000-W AM broadcast transmitter is being evaluated by means of a two-tone te transmitter is connected to a 50-Ω load and $m(t) = A_1 \cos \omega_1 t + A_1 \cos 2\omega_1 t$ $f_1 = 500$ Hz. Assume that a perfect AM signal is generated.

**(a)** Evaluate the complex envelope for the AM signal in terms of $A_1$ and $\omega_1$.

**(b)** Determine the value of $A_1$ for 90% modulation.

**(c)** Find the values for the peak current and average current into the 50-Ω load for the 90 ulation case.

**5-6** An AM transmitter uses a two-quadrant multiplier so that the transmitted signal is descr (5-7). Assume that the transmitter is modulated by $m(t) = A_m \cos \omega_m t$, where $A_m$ is adj that 120% positive modulation is obtained. Evaluate the spectrum of this AM signal in $A_c$, $f_c$, and $f_m$. Sketch your result.

**5-7** A DSB-SC signal is modulated by $m(t) = \cos \omega_1 t + 2 \cos 2\omega_1 t$ where $\omega_1$ $f_1 = 500$ Hz, and $A_c = 1$.

**(a)** Write an expression for the DSB-SC signal and sketch a picture of this waveform.

**(b)** Evaluate and sketch the spectrum for this DSB-SC signal.

**(c)** Find the value of the average (normalized) power

**(d)** Find the value of the PEP (normalized).

**5-8** Assume that transmitting circuitry restricts the modulated output signal to a certain pe say $A_p$, because of power-supply voltages that are used and the peak voltage and curre of the components. If a DSB-SC signal with a peak value of $A_p$ is generated by this circ that the sideband power of this DSB-SC signal is four times the sideband power of a ble AM signal having the same peak value, $A_p$, that could also be generated by this

**5-9** A DSB-SC signal can be generated from two AM signals as shown in Fig. P5-9. Usi matics to describe signals at each point on the figure prove that the output is a DSB-

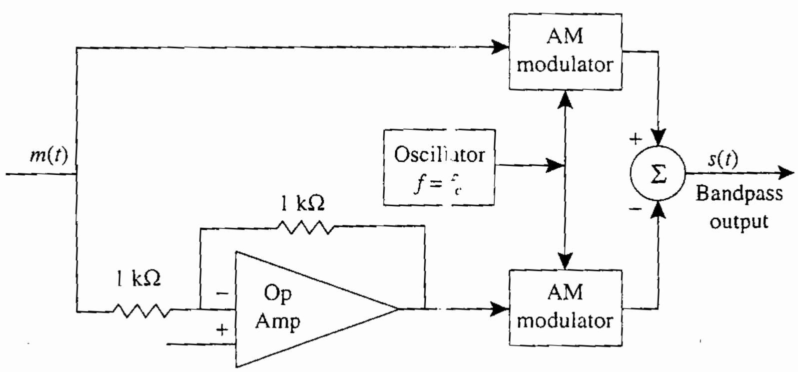



**Figure P5-9**

**5-10** Show that the complex envelope $g(t) = m(t) - j\hat{m}(t)$ produces a lower SSB sig that $m(t)$ is a real signal.

**5-11** Show that the impulse response of a −90° phase shift network (i.e., a Hilbert tr $1/\pi t$. *Hint:*

$$H(f) = \lim_{\substack{\alpha \to 0 \\ \alpha > 0}} \begin{cases} -je^{-\alpha f}, & f > 0 \\ je^{\alpha f}, & f < 0 \end{cases}$$

Chap. 5 Problems 371

**5-12** SSB signals can be generated by the phasing method, Fig. 5-5a; the filter method, Fig. 5-5b; or by the use of Weaver's method as shown in Fig. P5-12. For Weaver's method (Fig. P5-12) where $B$ is the bandwidth of $m(t)$:

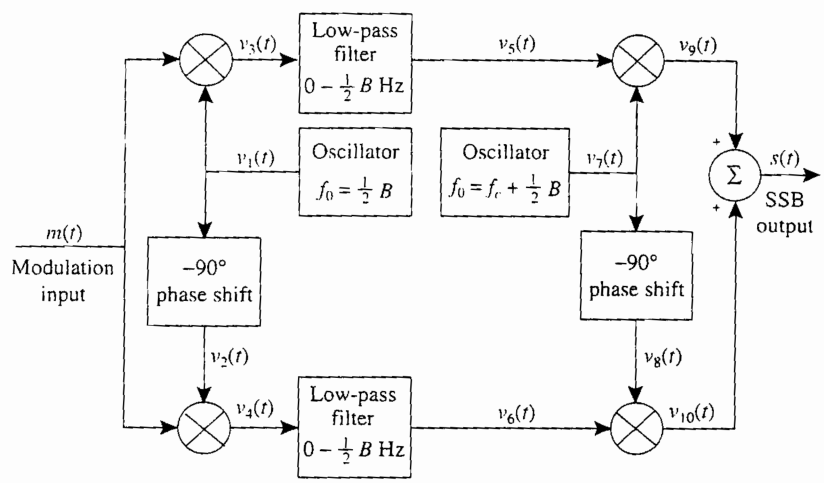



**Figure P5-12** Weaver's method for generating SSB.

(a) Find a mathematical expression that describes the waveform out of each block on the block diagram.
(b) Show that $s(t)$ is an SSB signal.

**5-13** An SSB-AM transmitter is modulated with a sinusoid $m(t) = 5 \cos \omega_1 t$, where $\omega_1 = 2\pi f_1$, $f_1 = 500$ Hz, and $A_c = 1$.
(a) Evaluate $\hat{m}(t)$.
(b) Find the expression for a lower SSB signal.
(c) Find the rms value of the SSB signal.
(d) Find the peak value of the SSB signal.
(e) Find the normalized average power of the SSB signal.
(f) Find the normalized PEP of the SSB signal.

**5-14** An SSB-AM transmitter is modulated by a rectangular pulse such that $m(t) = \Pi(t/T)$ and $A_c = 1$.
(a) Prove that

$$\hat{m}(t) = \frac{1}{\pi} \ln \left| \frac{2t + T}{2t - T} \right|$$

as given in Table A-7.
(b) Find an expression for the SSB-AM signal, $s(t)$, and sketch $s(t)$.
(c) Find the peak value of $s(t)$.

**5-15** For Prob. 5-14:
(a) Find the expression for the spectrum of a USSB-AM signal.
(b) Sketch the magnitude spectrum, $|S(f)|$.

**5-16** A USSB transmitter is modulated with the pulse

$$m(t) = \frac{\sin \pi a t}{\pi a t}$$

**(a)** Prove that

$$\hat{m}(t) = \frac{\sin^2[(\pi a/2)t]}{(\pi a/2)t}$$

**(b)** Plot the corresponding USSB signal waveform for the case of $A_c = 1$, $a = 2$, and $f_c = 20$ Hz.

**5-17** A USSB-AM signal is modulated by a rectangular pulse train:

$$m(t) = \sum_{n=-\infty}^{\infty} \Pi[(t - nT_0)/T]$$

where $T_0 = 2T$.

**(a)** Find the expression for the spectrum of the SSB-AM signal.

**(b)** Sketch the magnitude spectrum, $|S(f)|$.

**5-18** A phasing-type SSB-AM detector is shown in Fig. P5-18. This circuit is attached to the IF output of a conventional superheterodyne receiver to provide SSB reception.

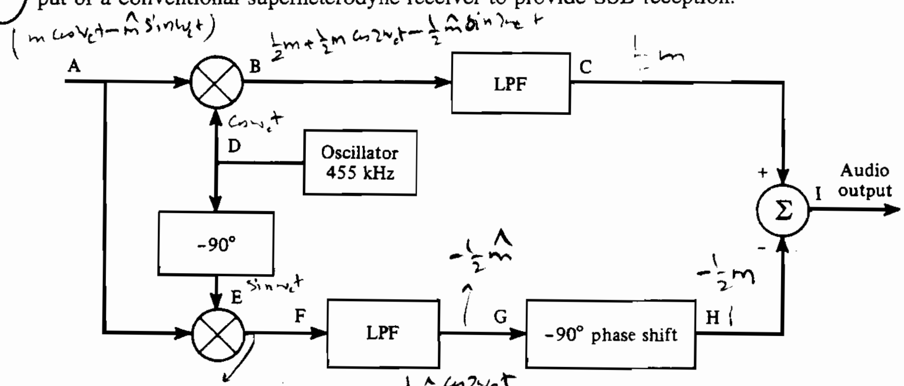



**Figure P5-18**

**(a)** Determine whether this detector is sensitive to LSSB or USSB signals. How would the detector be changed to receive SSB signals with alternate (opposite type of) sidebands?

**(b)** Assume that the signal at point $A$ is a USSB signal with $f_c = 455$ kHz. Find the mathematical expressions for the signals at points $B$ through $I$.

**(c)** Repeat part (b) for the case of an LSSB-AM signal at point $A$.

**(d)** Discuss the IF and LP filter requirements if the SSB signal at point $A$ has a 3-kHz bandwidth.

**5-19** Can a Costas loop, as shown in Fig. 5-3, be used to demodulate an SSB-AM signal? Demonstrate that your answer is correct by using mathematics.

**5-20** A modulated signal is described by the equation

$$s(t) = 10\cos[(2\pi \times 10^8)t + 10\cos(2\pi \times 10^3 t)]$$

Find each of the following.

(a) Percentage of AM.
(b) Normalized power of the modulated signal.
(c) Maximum phase deviation.
(d) Maximum frequency deviation.

**5-21** A sinusoidal signal, $m(t) = \cos 2\pi f_m t$, is the input to an angle-modulated transmitter where the carrier frequency is $f_c = 1$ Hz and $f_m = f_c/4$.
(a) Plot $m(t)$ and the corresponding PM signal where $D_p = \pi$.
(b) Plot $m(t)$ and the corresponding FM signal where $D_f = \pi$.

**5-22** A sinusoidal modulating waveform of amplitude 4 V and a frequency of 1 kHz is applied to an FM exciter that has a modulator gain of 50 Hz/V.
(a) What is the peak frequency deviation?
(b) What is the modulation index?

**5-23** An FM signal has sinusoidal modulation with a frequency of $f_m = 15$ kHz and modulation index of $\beta = 2.0$.
(a) Find the transmission bandwidth using Carson's rule.
(b) What percentage of the total FM signal power lies within the Carson rule bandwidth?

**5-24** An FM transmitter has a block diagram as shown in Fig. P5-24. The audio frequency response is flat over the 20-Hz to 15-kHz audio band. The FM output signal is to have a carrier frequency of 103.7 MHz and a peak deviation of 75 kHz.

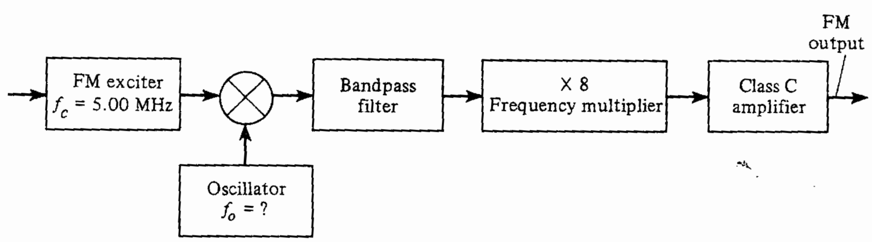



**Figure P5-24**

(a) Find the bandwidth and center frequency required for the bandpass filter.
(b) Calculate the frequency $f_0$ of the oscillator.
(c) What is the required peak deviation capability of the FM exciter?

**5-25** Analyze the performance of the FM circuit of Fig. 5-8b. Assume that the voltage appearing across the reversed-biased diodes, which provide the voltage variable capacitance, is $v(t) = 5 + 0.05m(t)$, where the modulating signal is a test tone, $m(t) = \cos \omega_1 t$, $\omega_1 = 2\pi f_1$, and $f_1 = 1$ kHz. The capacitance of each of the biased diodes is $C_d = 100/\sqrt{1 + 2v(t)}$ pF. Assume that $C_0 = 180$ pF and that $L$ is chosen to resonate at 5 MHz.
(a) Find the value of $L$.
(b) Show that the resulting oscillator signal is an FM signal. For convenience, assume that the peak level of the oscillator signal is 10 V. Find the parameter $D_f$.

**5-26** A modulated RF waveform is given by $500 \cos[\omega_c t + 20 \cos \omega_1 t]$, where $\omega_1 = 2\pi f_1$, $f_1 = 1$ kHz, and $f_c = 100$ MHz.
(a) If the phase deviation constant is 100 rad/V, find the mathematical expression for the corresponding phase modulation voltage $m(t)$. What is its peak value and its frequency?
(b) If the frequency deviation constant is $1 \times 10^6$ rad/V-s, find the mathematical expression for the corresponding FM voltage, $m(t)$. What is its peak value and its frequency?
(c) If the RF waveform appears across a 50-Ω load, determine the average power and the PEP.

**5-27** Given the FM signal $s(t) = 10 \cos [\omega_c t + 100 \int_{-\infty}^{t} m(\sigma)\, d\sigma]$, where $m(t)$ is a polar squar wave signal with a duty cycle of 50%, a period of 1 s, and a peak value of 5 V.

**(a)** Sketch the instantaneous frequency waveform and the waveform of the corresponding F signal (see Fig. 5-9).

**(b)** Plot the phase deviation $\theta(t)$ as a function of time.

**(c)** Evaluate the peak frequency deviation.

**5-28** A carrier $s(t) = 100 \cos(2\pi \times 10^9 t)$ of an FM transmitter is modulated with a tone signal. F this transmitter a 1-V (rms) tone produces a deviation of 30 kHz. Determine the amplitude a frequency of all FM signal components (spectral lines) that are greater than 1% of the unmo ulated carrier amplitude for the following modulating signals

**(a)** $m(t) = 2.5 \cos(3\pi \times 10^4 t)$.

**(b)** $m(t) = 1 \cos(6\pi \times 10^4 t)$.

**5-29** Referring to (5-58), show that

$$J_{-n}(\beta) = (-1)^n J_n(\beta)$$

**5-30** Consider an FM exciter with the output $s(t) = 100 \cos[2\pi 1000t + \theta(t)]$. The modulation $m(t) = 5 \cos(2\pi 8t)$ and the modulation gain of the exciter is 8 Hz/V. The FM output signal passed through an ideal (brickwall) bandpass filter which has a center frequency of 1000 Hz bandwidth of 56 Hz, and a gain of unity. Determine the normalized average power:

**(a)** At the bandpass filter input.

**(b)** At the bandpass filter output.

**5-31** A 1-kHz sinusoidal signal phase modulates a carrier at 146.52 MHz with a peak phase dev tion of 45°. Evaluate the exact magnitude spectra of the PM signal if $A_c = 1$. Sketch your sult. Using Carson's rule, evaluate the approximate bandwidth of the PM signal and see if i a reasonable number when compared with your spectral plot.

**5-32** A 1-kHz sinusoidal signal frequency modulates a carrier at 146.52 MHz with a peak deviat of 5 kHz. Evaluate the exact magnitude spectra of the FM signal if $A_c = 1$. Sketch your res Using Carson's rule, evaluate the approximate bandwidth of the FM signal and see if it is a r sonable number when compared with your spectral plot.

**5-33** The calibration of a frequency deviation monitor is to be verified by using a Bessel function t An FM test signal with a calculated frequency deviation is generated by frequency modulat a sine wave onto a carrier. Assume that the sine wave has a frequency of 2 kHz and that amplitude of the sine wave is slowly increased from zero until the discrete carrier term (at of the FM signal reduces to zero, as observed on a spectrum analyzer. What is the peak quency deviation of the FM test signal when the discrete carrier term is zero? Suppose that amplitude of the sine wave is increased further until this discrete carrier term appears, reac a maximum, and then disappears again. What is the peak frequency deviation of the FM test nal now?

**5-34** A frequency modulator has a modulator gain of 10 Hz/V and the modulating waveform is

$$m(t) = \begin{cases} 0, & t < 0 \\ 5, & 0 < t < 1 \\ 15, & 1 < t < 3 \\ 7, & 3 < t < 4 \\ 0, & 4 < t \end{cases}$$

**(a)** Plot the frequency deviation in hertz over the time interval $0 < t < 5$.

**(b)** Plot the phase deviation in radians over the time interval $0 < t < 5$.

### 3.7.2 Key solution

EE 443) HW # 7 Key page 1

Drill prob. # 3.2)

$s_{AM}(t) = A_c[1 + K_a m(t)]\cos 2\pi f_c t$

where $m(t) = A_m \cos 2\pi f_m t$ sinusoidal modulating wave

$s_{AM}(t) = A_c[1 + K_a A_m \cos 2\pi f_m t]\cos 2\pi f_c t \qquad f_c >> f_m$

$K_a A_m = 20\% = 0.2 \Rightarrow$

$s_{AM}(t) = A_c[1 + \overset{\mu}{0.2} \cos 2\pi f_m t]\cos 2\pi f_c t$

$= A_c \cos 2\pi f_c t + A_c \mu \cos 2\pi f_c t \cos 2\pi f_m t$

$= \underbrace{A_c \cos 2\pi f_c t}_{\text{carrier}} + \frac{A_c \mu}{2}\left\{\underbrace{\cos[2\pi(f_c + f_m)t]}_{U.SB} + \underbrace{\cos[2\pi(f_c - f_m)t]}_{L.SB}\right\}$

Thus: $P_c = \frac{A_c^2}{2}$

$P_{USB} = P_{LSB} = \frac{\left(\frac{A_c \mu}{2}\right)^2}{2} = \frac{A_c^2 \mu^2}{8} = \frac{A_c^2 (0.2)^2}{8} = \frac{A_c^2}{200}$

$= P_c \times 10^{-2}$

That is carrier has 98% of total power and each sideband has 1% the total power.

Drill prob. # 3.4)

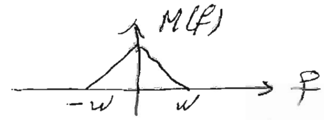
a) $v_1(t) = A_c \cos 2\pi f_c t + m(t) \qquad (1)$

$v_2(t) = a_1 v_1(t) + a_2 v_1^2(t) \qquad (2)$

$\Rightarrow v_2(t) = a_1\left(A_c \cos 2\pi f_c t + m(t)\right) + a_2\left(A_c \cos 2\pi f_c t + m(t)\right)^2$

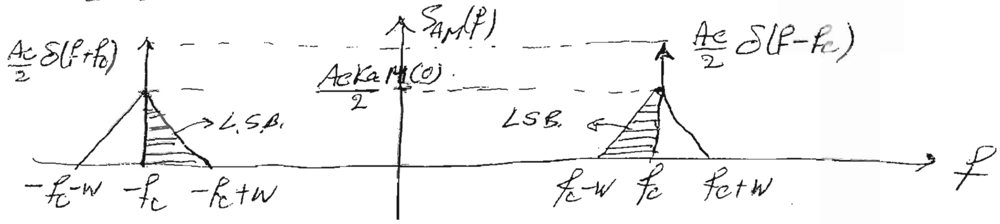
EE 443 HW # 7 page 2

Hill prob. # 3.3) Assume




In general $s_{AM}(t) = A_c[1 + K_a m(t)] \cos 2\pi f_c t$

$$S_{AM}(f) = \frac{A_c}{2}[\delta(f-f_c) + \delta(f+f_c)] + \frac{A_c K_a}{2}[M(f-f_c) + M(f+f_c)]$$




From the plot of $S_{AM}(f)$ we see that to avoid the overlapping of L.S.B frequencies it must

$f_c - W > 0 \Rightarrow f_c > W$

3.18)

Given: $\begin{cases} m(t) = 20\cos(2\pi t) \text{ volts} \\ c(t) = 50\cos(100\pi t) \text{ volts} \\ \mu = A_m K_a = 75\% = 0.75 \\ R = 100\,\Omega \end{cases}$

b) $s_{AM}(t) = A_c[1 + K_a m(t)]\cos 2\pi f_c t = 50[1 + \underbrace{K_a \cdot 20}_{\mu} \cos 2\pi t]\cos 100\pi t$

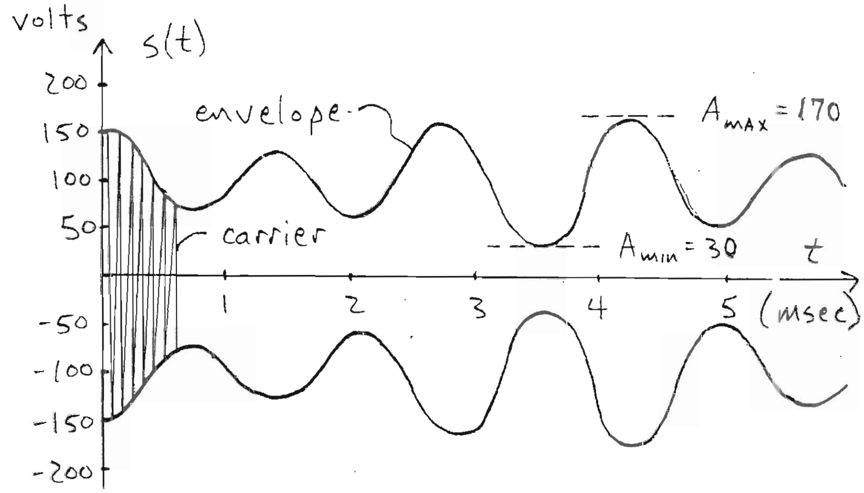
$= 50\cos(100\pi t) + 50 \times 0.75 \cos(100\pi t)\cos(2\pi t)$

$= 50\cos(100\pi t) + \frac{37.5}{2}\{\cos(102\pi t) + \cos(98\pi t)\}$

$$P_{tot} = \frac{(50)^2}{2R} + 2 \times \frac{\left(\frac{37.5}{2}\right)^2}{2R} = \frac{2500}{200} + \frac{1406.25}{400} = 16.0156 \text{ watts}$$

EE 443 HW #7 page 3

5-2. (a.) Cont'd

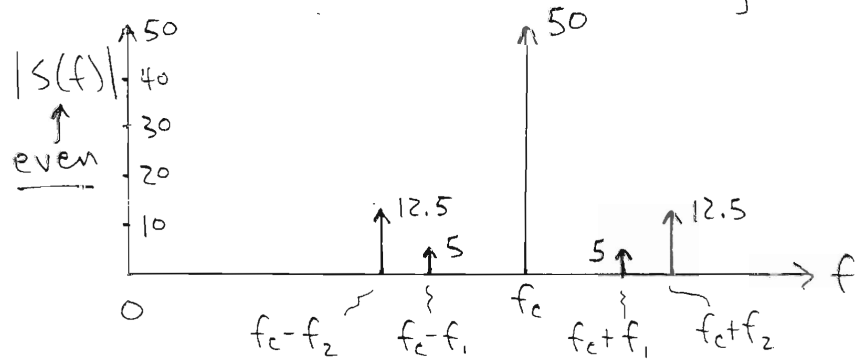



(b.) $\frac{170 - 30}{2(100)}(100) = \underline{\underline{70\ \%}}$ modulation

(c.) $G(f) = \delta(f) + M(f)$

$$= \delta(f) - j\frac{(0.2)}{2}\left[\delta(f-f_1) - \delta(f+f_1)\right] + \frac{(0.5)}{2}\left[\delta(f-f_2) + \delta(f+f_2)\right]$$

$$S(f) = \frac{A_c}{2}\left[G(f-f_c) + G^*(-f-f_c)\right]$$




160

## 3.8 HW 8

### Local contents

### 3.8.1 Questions

EE 443 chapt 3 HW# 8 page4

Drill 3.4)

$$v_2 = a_1 v_1(t) + a_2 v_1^2(t) \quad (1)$$

where, $v_1(t) = A_c \cos 2\pi f_c t + m(t) \quad (2)$

Subst. eq. (2) into eq (1)

$$v_2(t) = a_1\left[A_c \cos 2\pi f_c t + m(t)\right] + a_2\left[A_c \cos 2\pi f_c t + m(t)\right]^2$$

$$\Rightarrow v_2(t) = \underbrace{a_1 A_c\left[1 + \frac{2a_2}{a_1} m(t)\right]\cos 2\pi f_c t}_{\text{AM Signal}} + a_1 m(t) + a_2 m^2(t) + a_2 A_c^2 \underbrace{\cos^2(2\pi f_c t)}_{\frac{1}{2}[1+\cos 4\pi f_c t]}$$

The signal at the output of bandpass filter is:

$$v_0(t) = a_1 A_M\left[1 + \frac{2a_2}{a_1} m(t)\right]\cos 2\pi f_c t$$

which is an AM wave.

### 3.8.2 Key solution

EE 443 Hw # 8 Chapt (3) page 1

Drill Prob. # 3.4) $v_2(t) = a_1 v_1(t) + a_2 v_1^2(t)$ (1) , $v_1(t) = A_c \cos(2\pi f_c t) + m(t)$ (2)

$$v_2(t) = a_1 A_c \cos 2\pi f_c t + a_1 m(t) + a_2 A_c^2 \cos^2 2\pi f_c t + a_2 m^2(t) + 2 a_2 A_c\, m(t) \cos 2\pi f_c t \quad (3)$$

Assume $M(f)$ has the following form:


fig #1

Set $g(t) \cong m^2(t) = m(t) \cdot m(t)$

$\Rightarrow G(f) = M(f) \otimes M(f)$ The spectrum of $g(t) = m^2(t)$ will extend from $-2W$ to $2W$ Hz, for example would be

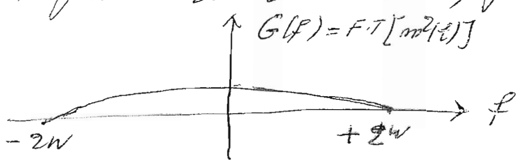



fig #2

Note: If you want to ~~g(t)~~ find $G(f)$, then you have to do $G(f) = \int_{-\infty}^{+\infty} M(\lambda) M(f-\lambda)\, d\lambda$. We are not interested to find the exact equation of $G(f)$, all we need to know is that the spectrum of $G(f) = F.T[m^2(t)]$ will extend from $-2W$ to $2W$ Hz.

Let us to take the F.T of eq (3) and plot it!

$$v_2(t) = a_1 A_c \cos 2\pi f_c t + a_1 m(t) + \frac{a_2 A_c^2}{2}\left[1 + \cos 4\pi f_c t\right] + a_2 m^2(t) + 2 a_2 A_c\, m(t) \cos 2\pi f_c t \quad (4)$$

$$\Rightarrow V_2(f) = \frac{a_1 A_c}{2}\left[\delta(f-f_c) + \delta(f+f_c)\right] + a_1 M(f) + \frac{a_2 A_c^2}{2}\delta(f) + \frac{a_2 A_c^2}{4}\left[\delta(f-2f_c) + \delta(f+2f_c)\right] + a_2 \underbrace{F.T[m^2(t)]}_{G(f)} + a_2 A_c\left[M(f-f_c) + M(f+f_c)\right] \quad (5)$$

EE 443 Hw # 8 Chapt. (3) page 2

The plot of eq.(5) is shown in figure # 3.

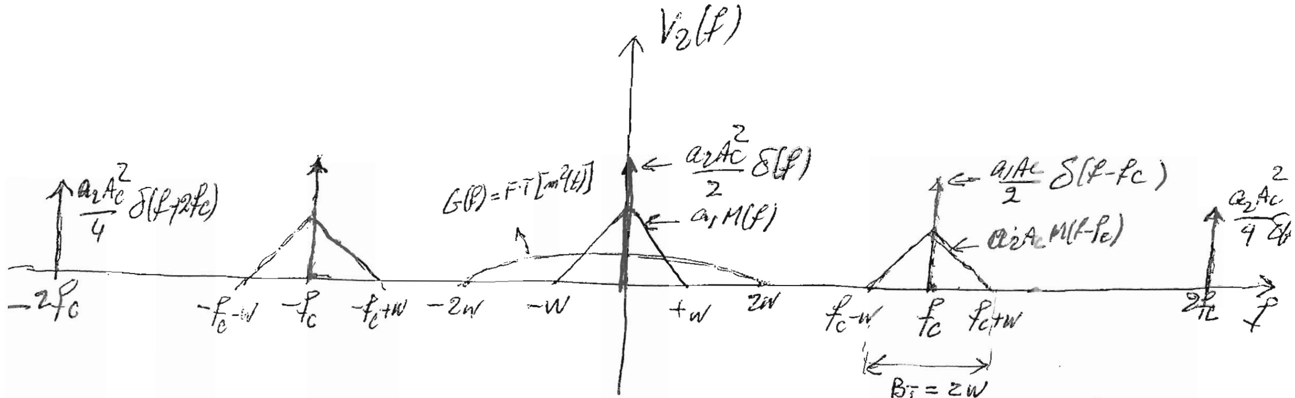



Fig # 3

Fig # 3 shows the spectral content of $V_2(t)$.

c) To extract the desired AM signal, use eq (4) and identify the AM signal:

$$V_2(t) = \underbrace{a_1 A_c\left[1 + \frac{2a_2}{a_1} m(t)\right]\cos 2\pi f_c t}_{\text{Desired AM Signal}} + \underbrace{\left(a_1 m(t) + a_2 m^2(t) + \frac{a_2 A_c^2}{2} + \frac{a_2 A_c^2}{2}\cos 4\pi f_c t\right)}_{\text{undesired component}} \quad (6)$$

A Bandpass filter centered at $f_c$ with total extend of $2W$ Hz, That is having a transfer function of:

$$H(f) = \text{rect}\left(\frac{f-f_c}{2W}\right) + \text{rec}\left(\frac{f+f_c}{2W}\right) \qquad (7)$$

will pass the desired signal (AM signal) and eliminated the unwanted components.

EE 443. Hw #8 Chapt 3 page 3

Using eq. (7) and figure #(3) we see that the required B.P.F must have a bandwidth of 2W Hz and centered at $f_c$, thus the cut-off frequencies of BPF are $f_c - W$ and $f_c + W$ Hz.

c) To avoide spectral overlapping of the desired signal (AM signal) with that of unwanted signals in $v_2(t)$, using figure # 3, we see that

1) $f_c - W \geq 2W \Rightarrow f_c \geq 3W$
2) $f_c + W \leq 2f_c \Rightarrow f_c \geq W$

Thus $f_c > 3W$

## 3.23

Assume m(t) with spectrum of

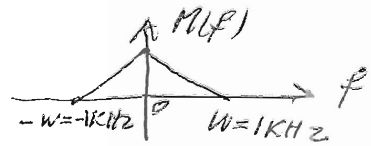
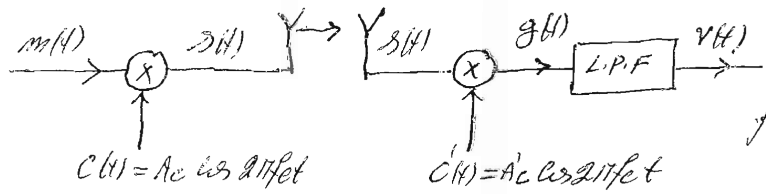



fig #1




fig#(2) Coherent detection of DSB-SC.

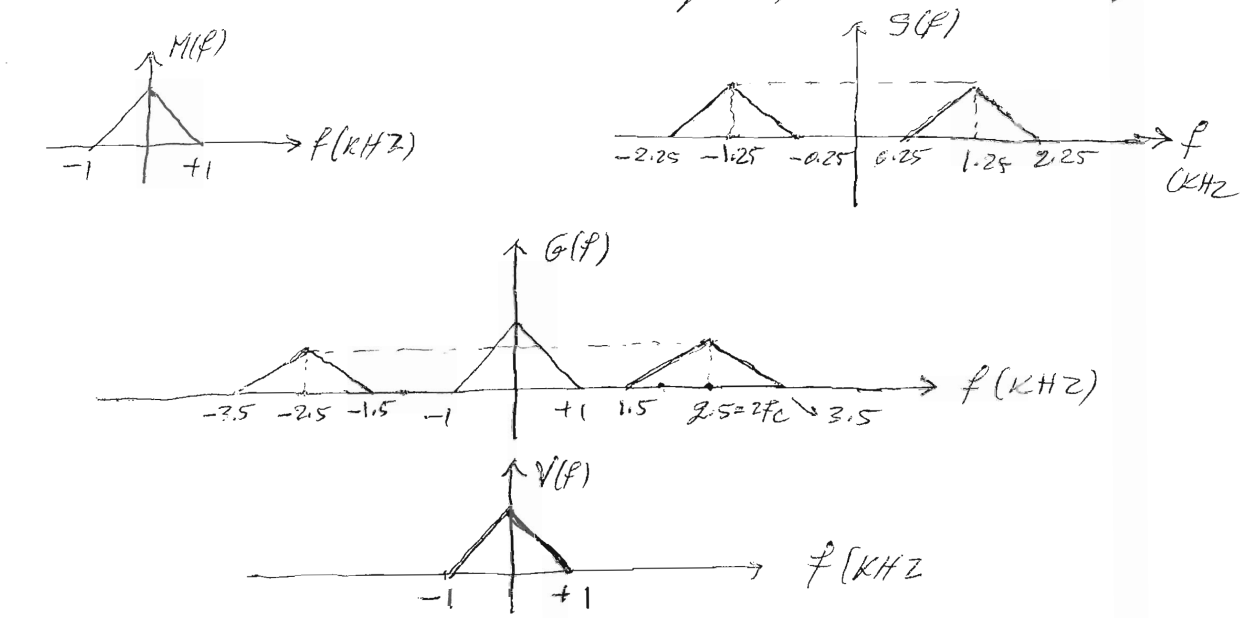
$$s(t) = m(t)\,c(t) = A_c\, m(t) \cos 2\pi f_c t \Rightarrow S(f) = \frac{A_c}{2}\left[M(f-f_c) + M(f+f_c)\right] \quad (1)$$

$$g(t) = s(t) \cdot c'(t) = A_c A'_c\, m(t) \cos^2 2\pi f_c t = \frac{A_c A'_c}{2} m(t)\left[1 + \cos 4\pi f_c t\right] \quad (2)$$

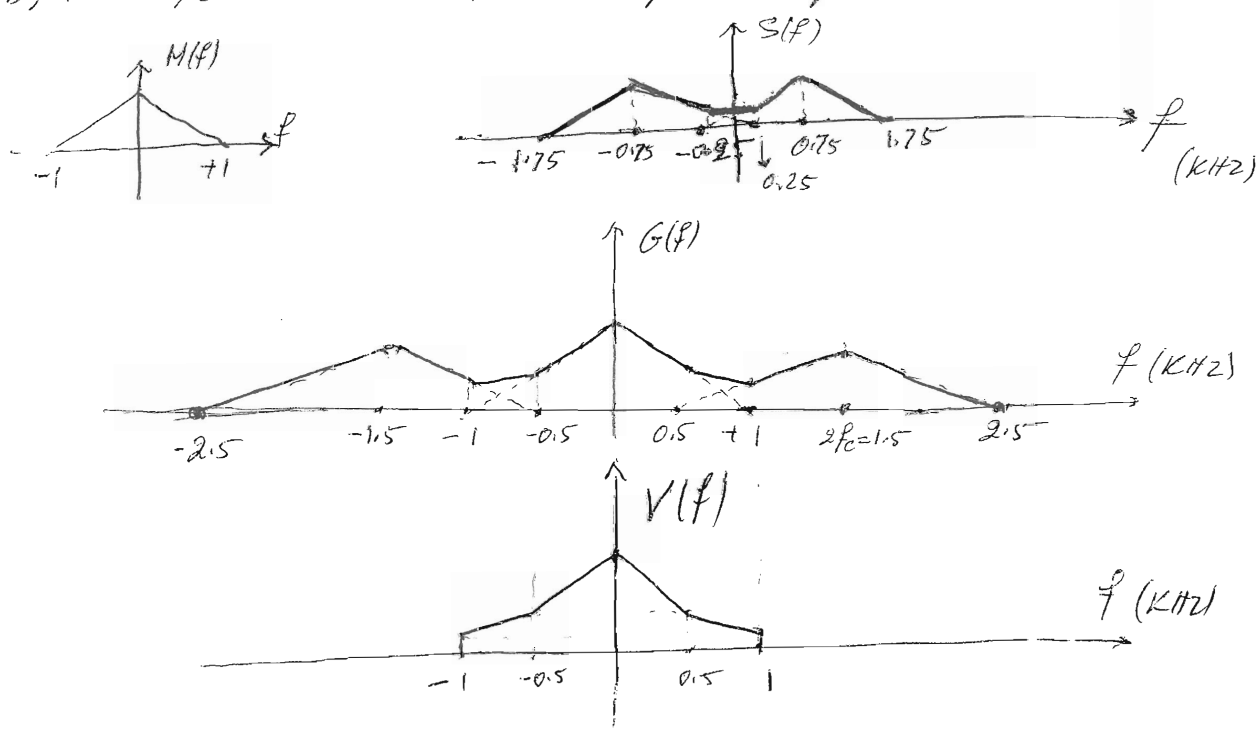
$$G(f) = \frac{A_c A'_c}{2} M(f) + \frac{A_c A'_c}{4}\left[M(f-2f_c) + M(f+2f_c)\right] \quad (3)$$

a) For $f_c = 1.25$ KHz, the spectrum of m(t), the spectrum of s(t) and the spectrum of v(t) (detector output) are:

EE 443 Hw # 8 Chapt-3) page (4)




b) For $f_c = 0.75$ KHZ, the respective spectra are:

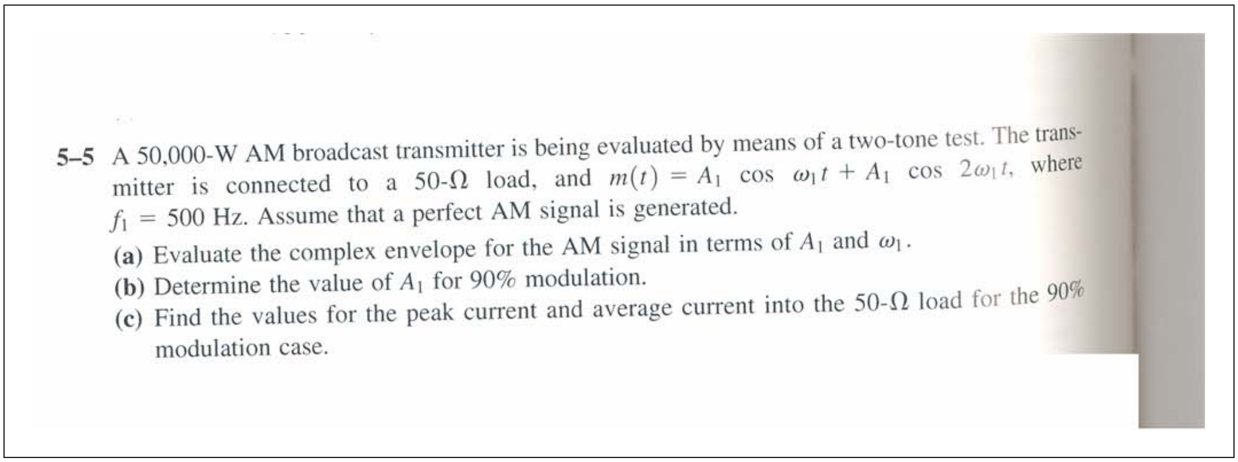


# 3.9 HW 9

## Local contents



## 3.9.1 Problem 5-5



**5–5** A 50,000-W AM broadcast transmitter is being evaluated by means of a two-tone test. The transmitter is connected to a 50-Ω load, and $m(t) = A_1 \cos \omega_1 t + A_1 \cos 2\omega_1 t$, where $f_1 = 500$ Hz. Assume that a perfect AM signal is generated.

**(a)** Evaluate the complex envelope for the AM signal in terms of $A_1$ and $\omega_1$.

**(b)** Determine the value of $A_1$ for 90% modulation.

**(c)** Find the values for the peak current and average current into the 50-Ω load for the 90% modulation case.

Figure 3.18: the Problem statement

#### 3.9.1.1 part(a)

$$s(t) = \overbrace{A_c\left(1 + k_a m(t)\right)}^{\text{in-phase component}} \cos \omega_c t$$

Assume $k_a = 1$ in this problem. $m(t) = A_1 (\cos \omega_1 t + \cos 2\omega_1 t)$, then $s(t)$ becomes

$$s(t) = \overbrace{A_c\left(1 + A_1\left(\cos \omega_1 t + \cos 2\omega_1 t\right)\right)}^{\text{in-phase component}} \cos \omega_c t \tag{1}$$

But $s(t)$ can be written as

$$s(t) = s_I(t) \cos \omega_c t - s_Q(t) \sin \omega_c t \tag{2}$$

Where $s_I(t)$ is the inphase component and $s_Q(t)$ is the quadrature component of $s(t)$. Compare (1) to (2), we see that

$$s_I(t) = A_c\left[1 + A_1\left(\cos \omega_1 t + \cos 2\omega_1 t\right)\right]$$
$$s_Q(t) = 0$$

Now, the complex envelope $\tilde{s}(t)$of $s(t)$ is given by

$$\tilde{s}(t) = s_I(t) + js_Q(t)$$

Hence replacing the value found for $s_I(t)$ and $s_Q(t)$ we obtain

$$\tilde{s}(t) = A_c[1 + A_1(\cos\omega_1 t + \cos 2\omega_1 t)] \tag{3}$$

Now, we can find $A_c$ since the average power in the carrier signal is given as 50000 watt as follows

$$P_{\text{av\_carrier}} = \frac{A_c^2}{2(50)} = 50000$$

Hence

$$A_c = \sqrt{100 \times 50000} = 2236.1\text{volt}$$

Then (3) becomes

$$\tilde{s}(t) = 2236.1[1 + A_1(\cos\omega_1 t + \cos 2\omega_1 t)] \tag{4}$$

The above is the complex envelope in terms of $A_1$ and $\omega_1$ only as required to show.

#### 3.9.1.2 part(b)

$$\mu = \frac{A_{\max} - A_{\min}}{A_{\max} + A_{\min}} \tag{5}$$

Need to find angle at which $\cos\omega_1 t + \cos 2\omega_1 t$ is Max and at which it is min. then Let $\Delta = \cos\omega_1 t + \cos 2\omega_1 t$

We see that when $\omega_1 t = 2\pi$, then $\Delta = 1 + 1 = 2$, hence

$$A_{\max} = A_c(1 + 2A_1)$$

Need to find $A_{\min}$ hence we need to find $\Delta_{\min.}$ For this case we must use calculus as it is not obvious where this is minimum

$$\begin{aligned}
\frac{\partial\Delta}{\partial t} &= 0 = -\omega_1 \sin\omega_1 t - 2\omega_1 \sin 2\omega_1 t \\
0 &= -\omega_1 \sin\omega_1 t - 2\omega_1(2\sin(\omega_1 t)\cos(\omega_1 t)) \\
&= -\omega_1 \sin\omega_1 t - 4\omega_1 \sin(\omega_1 t)\cos(\omega_1 t) \\
\frac{-1}{4} &= \cos(\omega_1 t)
\end{aligned}$$

Hence $\omega_1 t = \cos^{-1}\left(\frac{-1}{4}\right) \to \omega_1 t = 104.477^0$ (using calculator). hence

$$\begin{aligned}
\Delta_{\min} &= \cos\left(104.477^0\right) + \cos\left(2 \times 104.477^0\right) \\
&= -0.2499 - 0.875 \\
&= -1.1249
\end{aligned}$$

Then $A_{\min} = A_c(1 - 1.1249A_1)$, so from (5) above

$$\begin{aligned}
\mu &= \frac{A_{\max} - A_{\min}}{A_{\max} + A_{\min}} \\
0.9 &= \frac{A_c(1+2A_1) - A_c(1-1.1249A_1)}{A_c(1+2A_1) + A_c(1-1.1249A_1)} \\
&= \frac{(1+2A_1) - (1-1.1249A_1)}{(1+2A_1) + (1-1.1249A_1)} \\
&= \frac{1+2A_1 - 1 + 1.1249A_1}{1+2A_1+1-1.1249A_1} \\
&= \frac{3.1249A_1}{2+0.8751A_1}
\end{aligned}$$

Hence

$$\begin{aligned}
1.8 + 0.9(0.8751A_1) - 3.9A_1 &= 0 \\
1.8 - 2.3A_1 &= 0
\end{aligned}$$

Then

$$A_1 = 0.770$$

#### 3.9.1.3 part(c)

Since

$$\begin{aligned}
A_{\max} &= A_c(1+2A_1) \\
&= 2236.1(1 + 2\times 0.77012) \\
&= 5680.2 \text{ volts}
\end{aligned}$$

Then from Ohm's law, $V = RI$,

$$\begin{aligned}
I_{\max} &= \frac{V_{\max}}{R} \\
&= \frac{5680.2}{50} \\
&= 113.6 \text{ amps}
\end{aligned}$$

Since mean voltage is zero, then average current is zero.

### 3.9.2 Problem 5-8

**5–8** Assume that transmitting circuitry restricts the modulated output signal to a certain peak value, say, $A_p$, because of power-supply voltages that are used and because of the peak voltage and current ratings of the components. If a DSB-SC signal with a peak value of $A_p$ is generated by this circuit, show that the sideband power of this DSB-SC signal is four times the sideband power of a comparable AM signal having the same peak value $A_p$ that could also be generated by this circuit.

Figure 3.19: the Problem statement

answer For normal modulation, let

$$s_{am}(t) = A_c(1 + m(t))\cos\omega_c t$$

Maximum envelop is $2A_c$ (i.e. when $m_{\max}(t) = 1$), this means that $A_p = 2A_c$

But

$$s_{am}(t) = \overbrace{A_c\cos\omega_c t}^{\text{carrier}} + \overbrace{A_c m(t)\cos\omega_c t}^{\text{side band}}$$

So max of sideband is $A_c$ or $\frac{A_p}{2}$. Hence maximum power of sideband is $\frac{1}{2}\left(\frac{A_p}{2}\right)^2 = \frac{A_p^2}{8}$ and for DSB-SC, where now use $A_p$ in place of what we normally use $A_c$ then we obtain

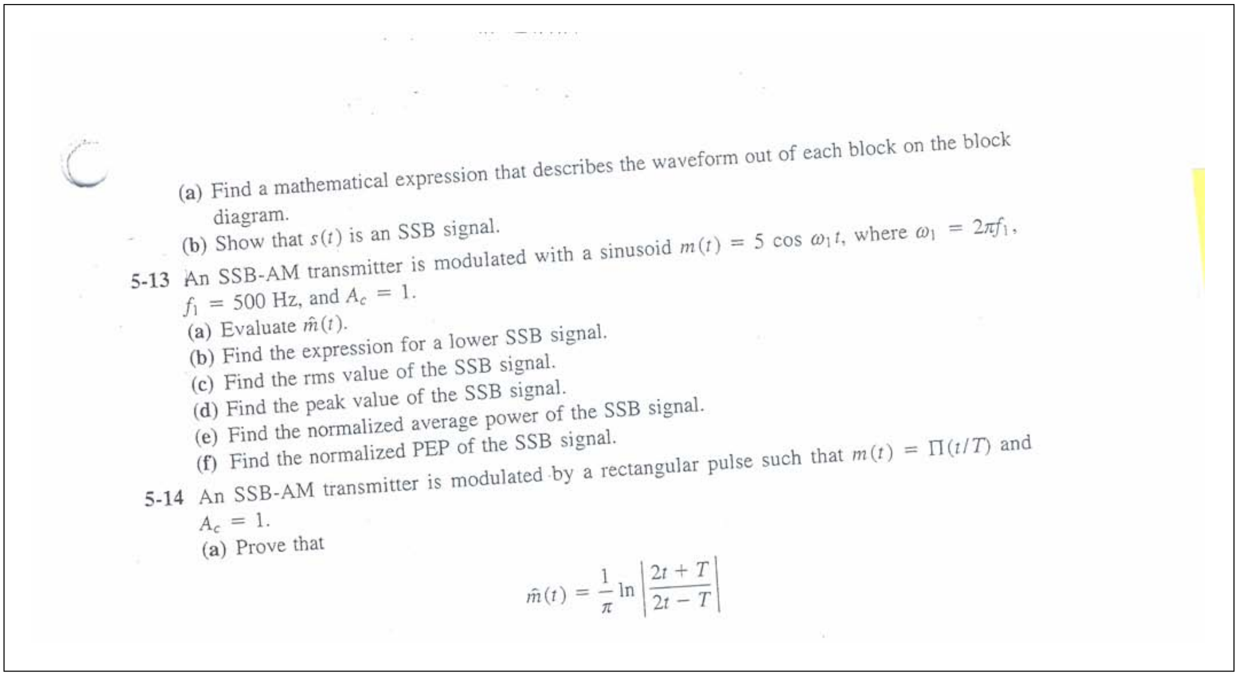
$$s(t) = A_p m(t)\cos\omega_c t$$

Hence maximum for sideband is $\frac{1}{2}A_p^2$

Hence we see that power of sideband of DSB-SC to the power of sideband of AM is

$$\frac{\frac{1}{2}A_p^2}{\frac{A_p^2}{8}} = 4$$

### 3.9.3 Problem 5-13


(a) Find a mathematical expression that describes the waveform out of each block on the block diagram.
(b) Show that $s(t)$ is an SSB signal.

**5-13** An SSB-AM transmitter is modulated with a sinusoid $m(t) = 5\cos\omega_1 t$, where $\omega_1 = 2\pi f_1$, $f_1 = 500$ Hz, and $A_c = 1$.
(a) Evaluate $\hat{m}(t)$.
(b) Find the expression for a lower SSB signal.
(c) Find the rms value of the SSB signal.
(d) Find the peak value of the SSB signal.
(e) Find the normalized average power of the SSB signal.
(f) Find the normalized PEP of the SSB signal.

**5-14** An SSB-AM transmitter is modulated by a rectangular pulse such that $m(t) = \Pi(t/T)$ and $A_c = 1$.
(a) Prove that

$$\hat{m}(t) = \frac{1}{\pi}\ln\left|\frac{2t+T}{2t-T}\right|$$

Figure 3.20: the Problem statement

#### 3.9.3.1 part(a)

$$m(t) = 5\cos\omega_1 t$$

$\hat{m}(t)$ is Hilbert transform of $m(t)$ defined as $\hat{m}(t) = \int\limits_{-\infty}^{\infty} m(\tau)\frac{1}{t-\tau}d\tau$. Or we can use the frequency approach where $\hat{m}(t) = \mathcal{F}^{-1}[-j\ sign(f)\ \ M(f)]$ where $M(f)$ is the Fourier transform of $m(t)$. We can carry out this easily, but since this is a phase 90 change, and $m(t)$ is a cosine function, then

$$\hat{m}(t) = 5\sin\omega_1 t$$

#### 3.9.3.2 part(b)

$$s_{SSB}(t) = A_c[m(t)\cos\omega_c t \mp \hat{m}(t)\sin\omega_c t]$$

Where the negative sign for upper sided band, and positive sign for the lower sided band, hence

$$\begin{aligned} s_{LSSB}(t) &= A_c[m(t)\cos\omega_c t + \hat{m}(t)\sin\omega_c t] \\ &= 5A_c[\cos\omega_1 t\cos\omega_c t + \sin\omega_1 t\sin\omega_c t] \\ &= 5A_c[\cos(\omega_c - \omega_1)t] \end{aligned}$$

We can plug in numerical values given

$$s_{LSSB}(t) = 5[\cos(\omega_c - \omega_1)t]$$

#### 3.9.3.3 Part(c)

To find the RMS value of the SSB, pick the above lower side band. First find $P_{av}$.

$$s_{LSSB}(t) = 5\left[\cos(\omega_1 - \omega_c)t\right]$$

Hence

$$\begin{aligned} RMS \text{ value of signal} &= \frac{5}{\sqrt{2}} \\ &= 3.535\,5 \text{ volt} \end{aligned}$$

#### 3.9.3.4 part(d)

Then maximum of $5\left[\cos(\omega_1 - \omega_c)t\right]$ is when $\cos(\omega_1 - \omega_c)t = 1$, hence

$$s_{LSSB_{\max}}(t) = 5\text{volt}$$

#### 3.9.3.5 part(e)

$$\begin{aligned} P_{av} &= \frac{1}{2}A_c^2 \\ &= \frac{1}{2} \times 25 \\ &= 12.5\text{watt} \end{aligned}$$

#### 3.9.3.6 Part(f)

$$\begin{aligned} PEP &= \frac{1}{2}s_{LSSB_{\max}}^2(t) \\ &= \frac{5^2}{2} \\ &= 12.5 \text{ watt} \end{aligned}$$

### 3.9.4 Problem 5-18

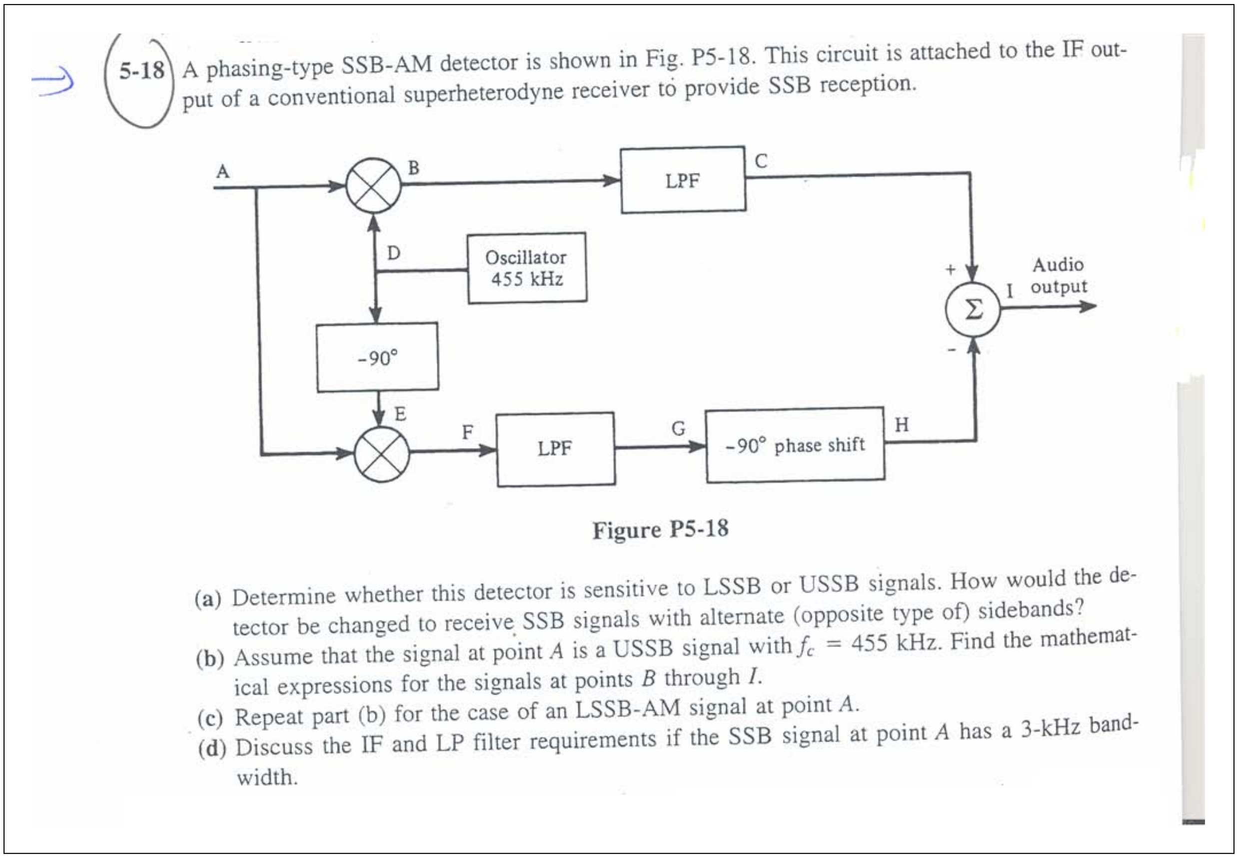

5-18 A phasing-type SSB-AM detector is shown in Fig. P5-18. This circuit is attached to the IF output of a conventional superheterodyne receiver to provide SSB reception.



Figure P5-18

(a) Determine whether this detector is sensitive to LSSB or USSB signals. How would the detector be changed to receive SSB signals with alternate (opposite type of) sidebands?
(b) Assume that the signal at point *A* is a USSB signal with $f_c$ = 455 kHz. Find the mathematical expressions for the signals at points *B* through *I*.
(c) Repeat part (b) for the case of an LSSB-AM signal at point *A*.
(d) Discuss the IF and LP filter requirements if the SSB signal at point *A* has a 3-kHz bandwidth.

Figure 3.21: the Problem statement

#### 3.9.4.1 part(a)

This is a detector for USSB (Upper side band). i.e.

$$s(t) = A_c (m(t)\cos\omega_c t - \hat{m}(t)\sin\omega_c t)$$

Note, I wrote $A_c$ and not $\frac{A_c}{2}$ in the above. As long this is a constant, it gives the same analysis.

The reason is because at point $H$ the signal is $-\frac{1}{2}m(t)$ and at the $C$ point the signal is $+\frac{1}{2}m(t)$ , hence due to subtraction at the audio output end we obtain $m(t)$. To receive LSSB, we should change the sign to positive at the audio output end.

#### 3.9.4.2 part(b)

$$s(t) = A_c (m(t)\cos\omega_c t - \hat{m}(t)\sin\omega_c t)$$

at point B

$$\begin{aligned}
s_B(t) &= s(t) * \overbrace{A_c^{'}\cos\omega_c t}^{\text{local oscillator}}\\
&= A_c^{'}A_c\left(m(t)\cos\omega_c t-\hat{m}(t)\sin\omega_c t\right)\cos\omega_c t\\
&= A_c^{'}A_c\left(m(t)\cos^2\omega_c t-\hat{m}(t)\sin\omega_c t\cos\omega_c t\right)\\
&= A_c^{'}A_c\left(m(t)\left(\frac{1}{2}+\frac{1}{2}\cos 2\omega_c t\right)-\frac{1}{2}\hat{m}(t)\sin 2\omega_c t\right)\\
&= \overbrace{\frac{A_c^{'}A_c}{2}m(t)}^{\text{low pass}}+\overbrace{\frac{A_c^{'}A_c}{2}m(t)\cos 2\omega_c t}^{\text{high pass}}-\overbrace{\frac{A_c^{'}A_c}{2}\hat{m}(t)\sin 2\omega_c t}^{\text{high pass}}
\end{aligned}$$

at point C, after LPF we obtain

$$s_c(t) = A_c^{'}A_c\frac{m(t)}{2}$$

at point F we have

$$\begin{aligned}
s_f(t) &= s(t)\,A_c^{'}\sin\omega_c t\\
&= A_c^{'}A_c\left(m(t)\cos\omega_c t-\hat{m}(t)\sin\omega_c t\right)\sin\omega_c t\\
&= A_c^{'}A_c\left(m(t)\cos(\omega_c t)\sin(\omega_c t)-\hat{m}(t)\sin^2\omega_c t\right)\\
&= A_c^{'}A_c\left(m(t)\frac{1}{2}\sin(2\omega_c t)-\hat{m}(t)\left(\frac{1}{2}-\frac{1}{2}\cos 2\omega_c t\right)\right)\\
&= \frac{A_c^{'}A_c}{2}\left(m(t)\sin(2\omega_c t)-\hat{m}(t)\left(1-\cos 2\omega_c t\right)\right)
\end{aligned}$$

at point G after LPF

$$s_g(t) = -\frac{A_c^{'}A_c}{2}\hat{m}(t)$$

at point H after $-90^0$ phase shift

$$s_h(t) = +\frac{A_c^{'}A_c}{2}m(t)$$

at point $I$, we sum $s_h(t)$ and $s_c(t)$, hence $s_i(t) = A_c^{'}A_c\frac{m(t)}{2}+\frac{A_c^{'}A_c}{2}m(t) = A_c^{'}A_c m(t)$

#### 3.9.4.3 Part(c)

$$s(t) = A_c\left(m(t)\cos\omega_c t+\hat{m}(t)\sin\omega_c t\right)$$

This the same as part (b), except now since there is a sign difference, this carries all the way to point $I$, and then we obtain

$$s_i(t) = A_c^{'}A_c\frac{m(t)}{2}-\frac{A_c^{'}A_c}{2}m(t) = 0$$

This if this circuit is used as is to demodulate an LSSB AM signal, then the signal will be lost. So, instead of adding at point $I$ we should now subtract to counter the effect of the negative sign.

#### 3.9.4.4 part(d)

Since SSB has bandwidth of $3kHz$ then this means the width of upper (or lower) band is $3khz$. This means the signal has $3khz$ bandwidth. This diagram shows the LPF requirement

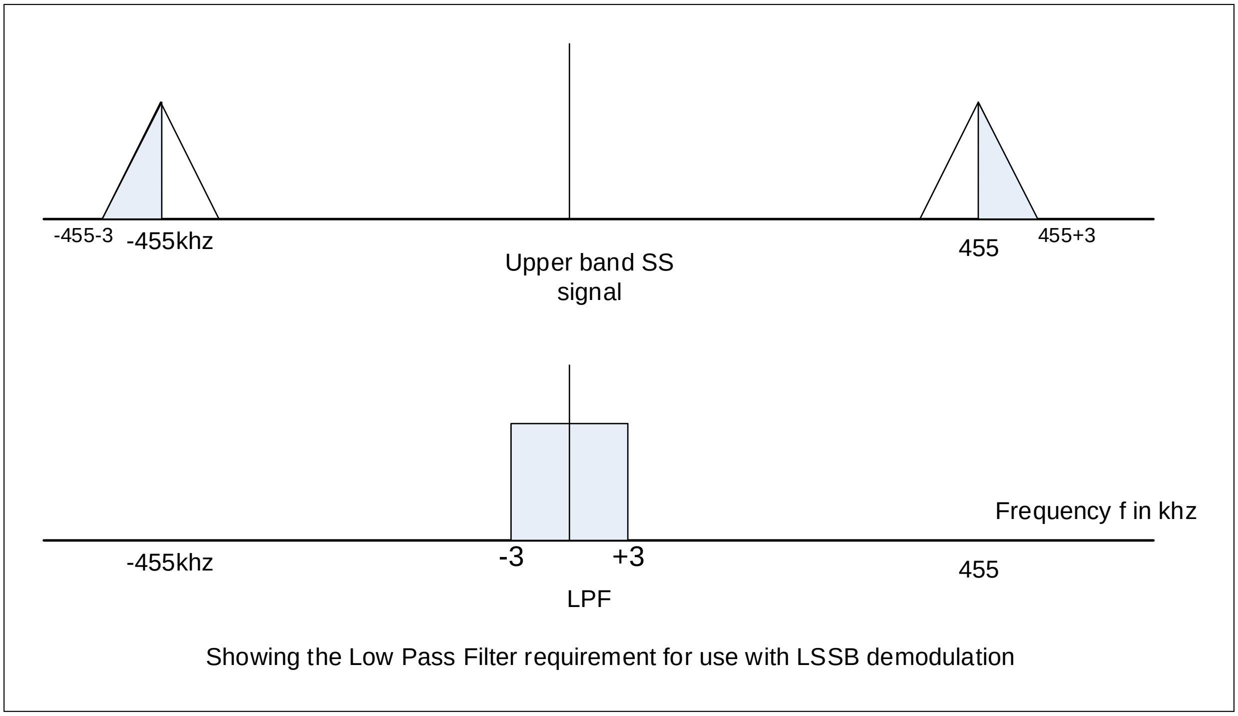



Figure 3.22: Low pass filter

Hence LPF is centered at zero frequency and have bandwidth of $3khz$ (may be make it a little over $3khz$ band width?)

The IF filter is centered at $455+\left(\frac{3}{2}\right)$ for the upper band of the positive band, and centered at $-455-\left(\frac{3}{2}\right)$ for the upper band of the negative band. (i.e. for the $USSB$).

For $LSSB$, IF should be centered at $455-\left(\frac{3}{2}\right)$ for the lower band of the positive band, and centered at $-455+\left(\frac{3}{2}\right)$ for the lower band of the negative band. (This works if there is a guard band around 455, small one, to make the design of IF possible).

### 3.9.5 Key solution

EE443 HW # 9. Key Nagel

**5-5.** (a.) $50{,}000 = \dfrac{A_c^2}{2(50)} \Rightarrow A_c = 2236\text{ V}$

$g(t) = A_c\{1 + m(t)\}$

$= 2236\{1 + A_1(\cos\omega_1 t + \cos 2\omega_1 t)\}$

(b.) to find $m(t)_{MIN}$: $x(\theta) = \cos\theta + \cos 2\theta$

$0 = \dfrac{dx(\theta)}{d\theta} = -\sin\theta - 2\sin 2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta \Rightarrow -\sin\theta = 4\sin\theta\cos\theta$

$\theta = 104.5^\circ$

$A_{max} = 2236\{1 + 2A_1\}$ $\qquad x(104.5^\circ) = -1.125$

$A_{MIN} = 2236\{1 - 1.125A_1\}$

$.90 = \dfrac{A_{max} - A_{min}}{2A_c} = \dfrac{3.125}{2}A_1 \Rightarrow A_1 = .576$

(c.) $A_{max} = 2236\{1 + 2(.576)\} = 4811.9$ volts

$I_{max} = \dfrac{A_{max}}{50} = 96.238$ Amps

$\langle s(t)\rangle = \langle 2236\{1 + .576(\cos\omega_1 t + \cos 2\omega_1 t)\} \cdot \cos\omega_c t\rangle$

$= 0$ for $\omega_c \gg \omega_1$

$\therefore I_{AV} = 0$ Amps

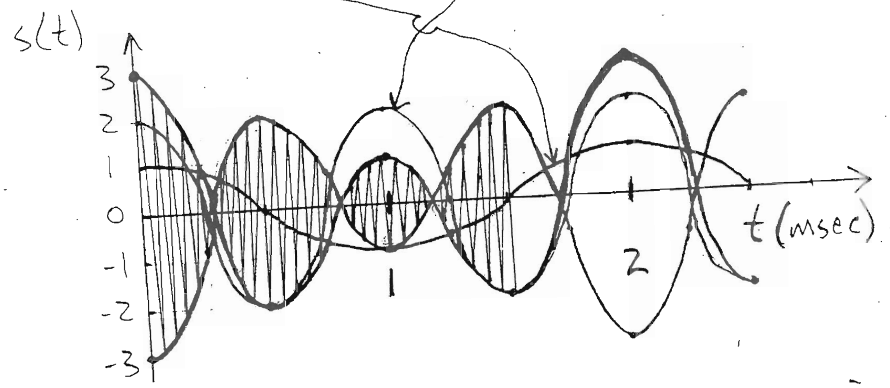
163

EE 443 HW # 9 page 2

5-7. (a.) DSB-SC $m(t) = \cos\omega_1 t + 2\cos 2\omega_1 t$

$$s(t) = [\cos\omega_1 t + 2\cos 2\omega_1 t]\cos\omega_c t$$

where $\omega_1 = 1000\pi$

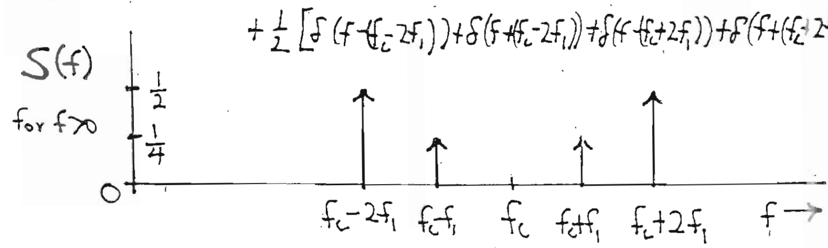



(b.) $s(t) = \frac{1}{2}[\cos(\omega_c - \omega_1)t + \cos(\omega_c + \omega_1)t]$

$+ \cos(\omega_c - 2\omega_1)t + \cos(\omega_c + 2\omega_1)t$

5-7 (b) Cont'd

$$S(f) = \mathcal{F}[s(t)] = \frac{1}{4}[\delta(f-(f_c-f_1)) + \delta(f+(f_c-f_1)) + \delta(f-(f_c+f_1)) + \delta(f+(f_c+f_1))]$$
$$+ \frac{1}{2}[\delta(f-(f_c-2f_1)) + \delta(f+(f_c-2f_1)) + \delta(f-(f_c+2f_1)) + \delta(f+(f_c+2f_1))]$$

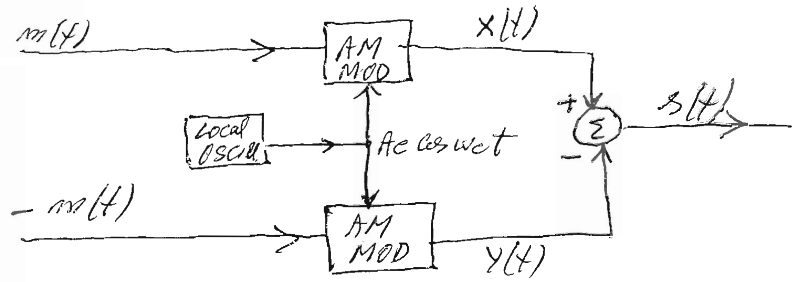



(c.) $P_{AV_{norm}} = \frac{1}{2}[(\frac{1}{2})^2 + (\frac{1}{2})^2 + (1)^2 + (1)^2] = \underline{1.25 W}$

(d.) $A_{max} = 3 \Rightarrow PEP_{norm} = \frac{(3)^2}{2} = \underline{4.5 W}$

EE 443 | Chapt, 2, 5 | HW # 8 | page

Prob. 5.9 ✓




where:

$x(t) = A_c[1 + K_a m(t)] \cos w_c t$

$y(t) = A_c[1 - K_a m(t)] \cos w_c t$

$s(t) = x(t) - y(t) = 2 A_c K_a m(t) \cos w_c t$

$s(t)$ is a DSBSC Signal.

Prob. # 5.13

$s(t) = \frac{A_c}{2} m(t) \cos 2\pi f_c t + \frac{A_c}{2} \hat{m}(t) \sin 2\pi f_c t$

Assume $A_c = 1$, $m(t) = 5 \cos 2\pi f_1 t$

$f_1 = 500\,Hz$.

a) $\hat{m}(t) = H \cdot T[m(t)]$

$\hat{M}(f) = -j\,\mathrm{sgn}(f) M(f) = -j\,\mathrm{sgn} f \left\{ \frac{5}{2} [\delta(f - f_1) + \delta(f + f_1)] \right\}$

$= \frac{-5j}{2} [\delta(f - f_1) - \delta(f + f_1)]$

$= \frac{5}{2j} [\delta(f - f_1) - \delta(f + f_1)]$

$\hat{m}(t) = F^{-1}T[\hat{M}(f)] = 5 \sin 2\pi f_1 t$

EE 443 | chapt 2, 5 | HW # 7 | page 6

prob # 5.13 cont'd)

b) $s_L(t) = \frac{5}{2} \cos w_1 t \cos w_c t + \frac{5}{2} \sin w_1 t \sin w_c t$ (1)

using $\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$

we have

$$s_L(t) = \frac{5}{2} \cos[(w_c - w_1)t] \quad (2)$$

c) $S_{rms} = \frac{S_{peak}}{\sqrt{2}} = \frac{5/2}{\sqrt{2}} = \frac{5}{2\sqrt{2}}$ volts

d) $S_{peak} = 5/2$ volts

e) $P_{av} = \frac{S_{peak}^2}{2} = \frac{(5/2)^2}{2} = \frac{25}{8}$ watts

f) PEP = ?

using eq. (1) find the envelope

$$a(t) = \sqrt{s_I^2(t) + s_Q^2(t)} = \sqrt{\left(\frac{5}{2}\right)^2 \cos^2 w_1 t + \left(\frac{5}{2}\right)^2 \sin^2 w_1 t}$$

$$a(t) = 5/2$$

$$PEP = \frac{1}{2}\left[\text{Max } a(t)\right]^2 = \frac{(5/2)^2}{2} = \frac{25}{8} \text{ watt}$$

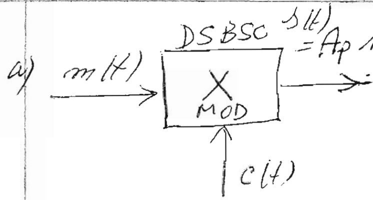
Note: $S_{rms}$ can be obtained from:

$$S_{rms}^2 = P_{av} = \frac{(5/2)^2}{2} = \frac{25}{8}$$

$$S_{rms} = \sqrt{P_{av}} = \frac{5}{2\sqrt{2}}$$

5.8)

a)

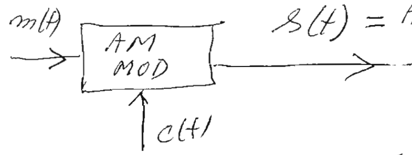



$P_{DSBSC} =$

$$P_{DSBSC} = \langle A_p^2\, m^2(t) \cos^2 w_c t \rangle = \langle A_p^2 \cos^2 w_c t \rangle \langle m^2(t) \rangle$$

$$= \frac{A_p^2}{2} \langle m^2(t) \rangle \qquad (1)$$

b)




$$s(t) = A_c [1 + km(t)] \cos w_c t$$

Assume $\max [m(t)] = 1$ To avoide envelope distortion $|m(t)| \le 1$

The problem stated that:

$$s(t)_{AM\,(peak)} = A_c [1 + \max(m(t))] = 2A_c \triangleq A_p$$

$$\Rightarrow A_c = \frac{A_p}{2}$$

Thus:

$$s(t)_{AM} = \frac{A_p}{2} [1 + m(t)] \cos w_c t$$

$$= \underbrace{\frac{A_p}{2} \cos w_c t}_{\text{Carrier}} + \underbrace{\frac{A_p}{2} m(t) \cos w_c t}_{\text{Sideband}}$$

$$P_{AM\,(SB)} = \left\langle \frac{A_p^2}{4} m^2(t) \cos^2 w_c t \right\rangle$$

$$= \frac{A_p^2}{4} \underbrace{\langle \cos^2 w_c t \rangle}_{1/2} \cdot \langle m^2(t) \rangle = \frac{A_p^2}{4 \times 2} \langle m^2(t) \rangle$$

eq (2)

Thus.

$$\frac{P_{DSBSC}}{P_{AM\,(SB)}} = \frac{\frac{A_p^2}{2} \langle m^2(t) \rangle}{\frac{A_p^2}{8} \langle m^2(t) \rangle} = 4 = 6\,dB$$

# 3.10 HW 10

## Local contents



### 3.10.1 Problem 3.24

**3.24** Consider a composite wave obtained by adding a noncoherent carrier $A_c \cos(2\pi f_c t + \phi)$ to a DSB-SC wave $\cos(2\pi f_c t)m(t)$. This composite wave is applied to an ideal envelope detector. Find the resulting detector output for

(a) $\phi = 0$

(b) $\phi \neq 0$ and $|m(t)| \ll A_c/2$

Figure 3.23: the Problem statement

$$s_1(t) = A_c \cos(\omega_c t + \phi)$$

DSB-SC signal is

$$s_2(t) = m(t) \cos(\omega_c t)$$

Hence by adding the above, we obtain

$$s(t) = m(t) \cos(\omega_c t) + A_c \cos(\omega_c t + \phi)$$

The above signal is applied to an ideal envelope detector. The output of an envelope detector is given by

$$a(t) = \sqrt{s_I^2(t) + s_Q^2(t)}$$

Since $s(t)$ is a bandpass signal, we need to first write it in the canonical form $s_I(t)\cos(\omega_c t) - s_Q(t)\sin(\omega_c t)$

Using $\cos(A+B) = \cos A \cos B - \sin A \sin B$, then we have

$$
\begin{aligned}
s\left(t\right) &= m\left(t\right)\cos\left(\omega_c t\right) + A_c\left[\cos\omega_c t\cos\phi - \sin\omega_c t\sin\phi\right] \\
&= \left[m\left(t\right) + A_c\cos\phi\right]\cos\left(\omega_c t\right) - A_c\sin\omega_c t\sin\phi
\end{aligned}
$$

Hence we see that

$$
\begin{aligned}
s_I\left(t\right) &= m\left(t\right) + A_c\cos\phi \\
s_Q\left(t\right) &= A_c\sin\phi
\end{aligned}
$$

Now we can start answering parts (a) and (b)

#### 3.10.1.1 Part(a)

When $\phi = 0$, then

$$
\begin{aligned}
s_I\left(t\right) &= m\left(t\right) + A_c \\
s_Q\left(t\right) &= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
a\left(t\right) &= \sqrt{\left[m\left(t\right) + A_c\right]^2 + 0^2} \\
&= m\left(t\right) + A_c
\end{aligned}
$$

### 3.10.2 Part(b)

When $\phi \neq 0$ and $\left|m\left(t\right)\right| << \frac{A_c}{2}$

$$
\begin{aligned}
a\left(t\right) &= \sqrt{\left[m\left(t\right) + A_c\right]^2 + \left[A_c\sin\phi\right]^2} \\
&= \sqrt{\left[m^2\left(t\right) + A_c^2 + 2A_c m\left(t\right)\right] + \left[A_c^2\sin^2\phi\right]}
\end{aligned}
$$

Since $\left|m\left(t\right)\right| << \frac{A_c}{2}$, then $m^2\left(t\right) + A_c^2 + 2A_c m\left(t\right) \simeq A_c^2$ hence

$$
\begin{aligned}
a\left(t\right) &\simeq \sqrt{A_c^2 + A_c^2\sin^2\phi} \\
&= A_c\sqrt{1 + \sin^2\phi}
\end{aligned}
$$

## 3.10.3 Problem 5.20

**5–20** A modulated signal is described by the equation

$$s(t) = 10\cos[(2\pi \times 10^8)t + 10\cos(2\pi \times 10^3 t)]$$

Find each of the following:
**(a)** Percentage of AM.
**(b)** Normalized power of the modulated signal.
**(c)** Maximum phase deviation.
**(d)** Maximum frequency deviation.

Figure 3.24: the Problem statement

### 3.10.3.1 Part(a)

An AM signal is $s(t) = A_c[1 + \mu\ m(t)]\cos(2\pi f_c t + \theta(t))$. Now compare this form with the one given above, which is $s(t) = A_c\cos(2\pi f_c t + \theta(t))$. We see that $\mu = 0$, i.e. no message source exist. Hence percentage of modulation is zero.

### 3.10.3.2 Part(b)

$$P_{av} = \frac{1}{2}A_c^2$$

But $A_c = 10$, hence

$$\begin{aligned} P_{av} &= \frac{100}{2} \\ &= 50\text{watt} \end{aligned}$$

### 3.10.3.3 Part(c)

From the general form for angle modulated signal

$$s(t) = \cos(\omega_c t + \theta(t))$$

Looking at

$$s(t) = A_c\cos\left(\overbrace{\overbrace{\left(2\pi \times 10^8\right)}^{2\pi f_c}t + \overbrace{10\cos\left(2\pi \times 10^3 t\right)}^{\theta(t)}}^{Total\ \ Phase}\right)$$

Phase deviation is

$$\theta(t) = 10\cos\left(2\pi \times 10^3 t\right)$$

Which is maximum when $\cos\left(2\pi \times 10^3 t\right) = 1$Hence maximum Phase deviation is 10 radians.

#### 3.10.3.4 part(d)

Now, we know that the instantenouse frequency $f_i$ is given by

$$\begin{aligned} f_i(t) &= \frac{1}{2\pi}\frac{d}{dt}(\text{total phase}) \\ &= \frac{1}{2\pi}\frac{d}{dt}\left[\omega_c t + \theta(t)\right] \\ &= \frac{1}{2\pi}\frac{d}{dt}\left[2\pi f_c t + 10\cos\left(2\pi \times 10^3 t\right)\right] \\ &= f_c - 10\left(10^3\right)\sin\left(2\pi \times 10^3 t\right) \end{aligned}$$

The deviation of frequency is the difference between $f_i$ and the carrier frequency $f_c$. Hence from the above we see that the frequency deviation is

$$\begin{aligned} \Delta f &= f_i - f_c \\ &= -10\left(10^3\right)\sin\left(2\pi \times 10^3 t\right) \end{aligned}$$

So, maximum $\Delta f$ occures when $\sin\left(2\pi \times 10^3 t\right) = -1$, hence

$$\max(\Delta f) = 10^4 \text{ Hz}$$

### 3.10.4 Problem 5.22

**5–22** A sinusoidal modulating waveform of amplitude 4 V and frequency 1 kHz is applied to an FM exciter that has a modulator gain of 50 Hz/V.
**(a)** What is the peak frequency deviation?
**(b)** What is the modulation index?

Figure 3.25: the Problem statement

The modulating waveform is $m(t)$ Hence (I am assuming it is cos since it said sinusoidal)

$$\begin{aligned} m(t) &= A_m \cos\left(2\pi f_m t\right) \\ &= 4\cos\left(2000\pi t\right) \end{aligned}$$

Since it is an FM signal, then

$$s(t) = A_c \cos\left[\overbrace{\omega_c t + 2\pi k_f \int_0^t m(x)\,dx}^{\theta(t)}\right]$$

Where $k_f$ is the frequency deviation constant in cycle per volt-second. The gain here means the frequency gain, which is the frequency deviation (deviation from the $f_c$ frequency). Let $\Delta f$ be the frequency deviation in Hz, then

$$\begin{aligned}
\Delta f &= f_i - f_c \\
&= \frac{1}{2\pi}\frac{d}{dt}\theta(t) \\
&= k_f m(t) \\
&= k_f\left[4\cos(2000\pi t)\right]
\end{aligned}$$

#### 3.10.4.1 Part(a)

max $\Delta f$ is

$$(\Delta f)_{\max} = 4k_f$$

But $k_f = 50$ hz/volt, hence

$$\begin{aligned}
(\Delta f)_{\max} &= 4 \times 50 \\
&= 200\text{hz}
\end{aligned}$$

#### 3.10.4.2 Part(b)

Modulation index

$$\begin{aligned}
\beta &= \frac{(\Delta f)_{\max}}{f_m} \\
&= \frac{200}{1000} \\
&= 0.2
\end{aligned}$$

## 3.10.5 Problem 5.24

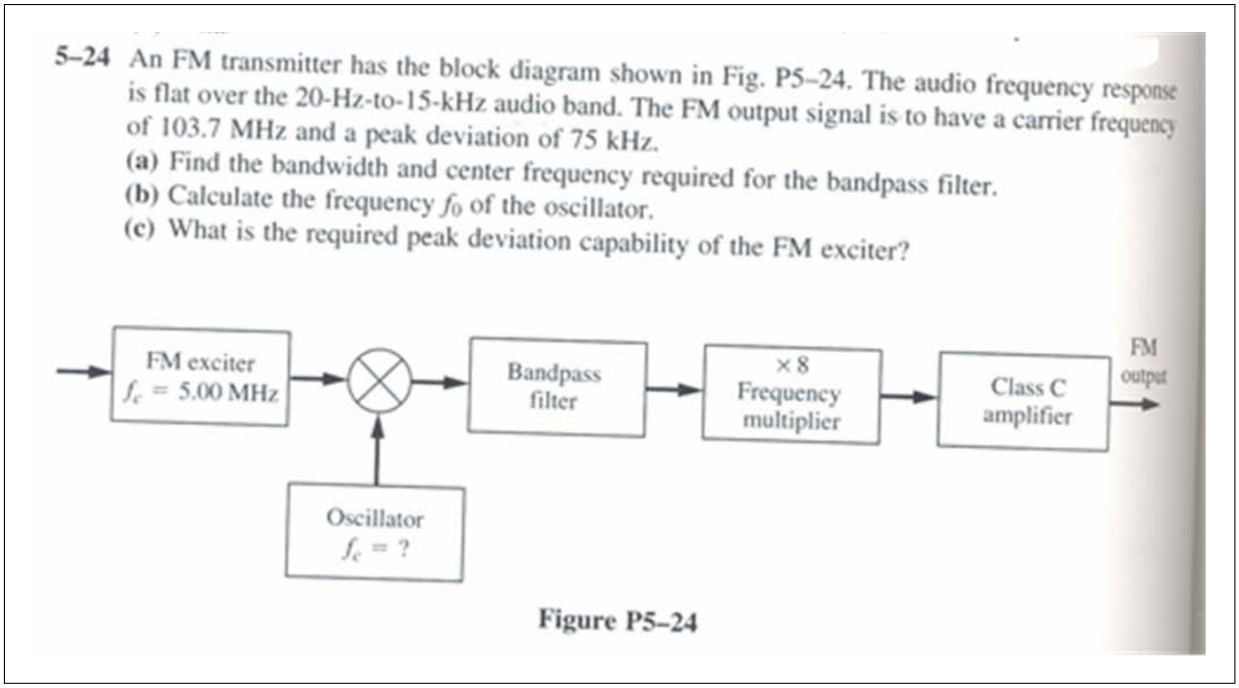

**5–24** An FM transmitter has the block diagram shown in Fig. P5–24. The audio frequency response is flat over the 20-Hz-to-15-kHz audio band. The FM output signal is to have a carrier frequency of 103.7 MHz and a peak deviation of 75 kHz.
**(a)** Find the bandwidth and center frequency required for the bandpass filter.
**(b)** Calculate the frequency $f_0$ of the oscillator.
**(c)** What is the required peak deviation capability of the FM exciter?



Figure 3.26: the Problem statement

$$s(t) = A_c \cos\left(2\pi f_c t + 2\pi k_f \int_0^t m(x)\,dx\right)$$

We are told the carrier frequency has $f_c = 103.7$ Mhz, but there is a multiplier of 8, and hence the center frequency of the bandpass filter must be $\frac{1}{8}$ of the carrier frequency. i.e.

center frequency of the bandpass filter is $\frac{1}{8}103.7 = \frac{103.7}{8} = 12.963$

Since peak deviation is $75khz$, which means the deviation from the central frequency has maximum of $75khz$, then

$$\frac{75}{8} = 9.375 \text{ khz}$$

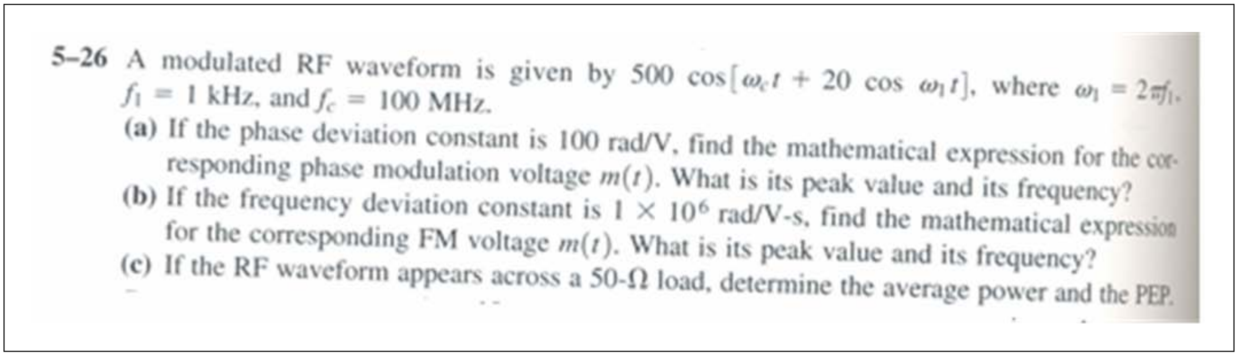
Hence bandwidth from center of frequency of bandwidth filter is 9.375 but we need to add frequency width of the audio which is $15000 - 20 = 14980$ Hz on both side, hence

Bandwidth of BPF is $9.375 \times 10^3 \pm 14980$

#### 3.10.5.1 Part (b)

To do

### 3.10.6 Problem 5.26



**5–26** A modulated RF waveform is given by 500 cos$[\omega_c t + 20 \cos \omega_1 t]$, where $\omega_1 = 2\pi f_1$, $f_1 = 1$ kHz, and $f_c = 100$ MHz.
**(a)** If the phase deviation constant is 100 rad/V, find the mathematical expression for the corresponding phase modulation voltage $m(t)$. What is its peak value and its frequency?
**(b)** If the frequency deviation constant is $1 \times 10^6$ rad/V-s, find the mathematical expression for the corresponding FM voltage $m(t)$. What is its peak value and its frequency?
**(c)** If the RF waveform appears across a 50-Ω load, determine the average power and the PEP.

Figure 3.27: the Problem statement

$$s(t) = A_c \cos(\omega_c t + 20 \cos \omega_1 t)$$

where $A_c = 500, f_1 = 1khz, f_c = 100Mhz$

#### 3.10.6.1 Part(a)

The general form of the above PM signal is

$$s(t) = A_c \cos\left(\omega_c t + \overbrace{k_p m(t)}^{\text{phase deviation}}\right)$$

Where $k_p m(t)$ is the phase deviation, and $k_p$ is the phase deviation constant in radians per volt. Hence we write

$$k_p m(t) = 20 \cos \omega_1 t$$

Then

$$m(t) = \frac{20 \cos \omega_1 t}{k_p}$$

But we are given that $k_p = 100$ rad/voltage and $f_1 = 1000hz$, then the above becomes

$$\begin{aligned} m(t) &= \frac{20 \cos(2000\pi t)}{100} \\ &= 0.2 \cos(2000\pi t) \end{aligned}$$

its frequency is 1 khz and its peak value is 0.2 volts

#### 3.10.6.2 Part(b)

The general form of the above FM signal is

$$s(t) = A_c \cos\left(\omega_c t + k_f \int_0^t m(x)\,dx\right)$$

Where $k_f$ is the frequency deviation constant in radians per volt-second

Hence

$$k_f \int_0^t m(x)\,dx = 20\cos\omega_1 t$$

Solve for $m(t)$ in the above, given that $k_f = 10^6$radians per volt-second, hence

$$\begin{aligned} k_f \int_0^t m(x)\,dx &= 20\cos\omega_1 t \\ \int_0^t m(x)\,dx &= \frac{20\cos(2000\pi t)}{10^6} \end{aligned}$$

Take derivative of both sides, we obtain

$$\begin{aligned} m(t) &= \frac{20}{10^6}\left[-\sin(2000\pi t)\times 2000\pi\right] \\ &= -\frac{20\times 2000\pi}{10^6}\sin(2000\pi t) \\ &= -0.126\sin(2000\pi t) \end{aligned}$$

Hence its peak value is 0.126 and its frequency is 1 khz

#### 3.10.6.3 Part(c)

$$\begin{aligned} P_{av} &= \frac{\langle s^2(t)\rangle}{50} \\ &= \frac{\frac{1}{2}A_c^2}{50} \\ &= \frac{500^2}{100} \\ &= 2500\text{watt} \end{aligned}$$

PEP is average power obtained if the complex envelope is held constant at its maximum values. i.e. (the normalized PEP) is

$$PEP = \frac{1}{2}\left[\max\left(\left|\tilde{s}\left(t\right)\right|\right)\right]^2$$

Since

$$\begin{aligned} s\left(t\right) &= A_c \cos\left(\omega_c t + 20\cos\omega_1 t\right) \\ &= A_c \left[\cos\omega_c t \cos\left(20\cos\omega_1 t\right) - \sin\omega_c t \sin\left(20\cos\omega_1 t\right)\right] \\ &= \overbrace{A_c \cos\left(20\cos\omega_1 t\right)}^{s_I(t)} \cos\omega_c t - \overbrace{A_c \sin\left(20\cos\omega_1 t\right)}^{s_Q(t)} \sin\omega_c t \end{aligned}$$

Hence

$$\begin{aligned} \tilde{s}\left(t\right) &= s_I\left(t\right) + j s_Q\left(t\right) \\ &= A_c \cos\left(20\cos\omega_1 t\right) + j A_c \sin\left(20\cos\omega_1 t\right) \end{aligned}$$

Then

$$\begin{aligned} \left|\tilde{s}\left(t\right)\right| &= \sqrt{\left[A_c \cos\left(20\cos\omega_1 t\right)\right]^2 + \left[A_c \sin\left(20\cos\omega_1 t\right)\right]^2} \\ &= A_c \sqrt{\cos^2\left(20\cos\omega_1 t\right) + \sin^2\left(20\cos\omega_1 t\right)} \\ &= A_c \end{aligned}$$

Hence the non-normalized PEP is

$$\begin{aligned} PEP &= \frac{\frac{1}{2}\left[A_c\right]^2}{50} \\ &= \frac{500^2}{100} \\ &= 2500\text{watt} \end{aligned}$$

ps. is there an easier or more direct way to find PEP than what I did? (assuming it is correct)

### 3.10.7 Key solution

EE 443 HW# 10 page1

and part of 9

**5-18.** (a.)

(A) $v_A(t) = m(t)\cos\omega_{if}t \mp \hat{m}(t)\sin\omega_{if}t$ (upper sign: USSB, lower sign: LSSB)

(D) $v_D(t) = \cos\omega_{if}t$

(B) $v_B(t) = v_A(t)v_D(t) = m(t)\cos^2\omega_{if}t \mp \hat{m}(t)\sin\omega_{if}t\cos\omega_{if}t$

$= \frac{m(t)}{2}(1+\cos 2\omega_{if}t) \mp \frac{\hat{m}(t)}{2}\sin 2\omega_{if}t$

(C) $v_C(t) = \frac{m(t)}{2}$

(E) $v_E(t) = \sin\omega_{IF}t$

(F) $v_F(t) = v_A(t)\,v_E(t)$

$= m(t)\sin\omega_{IF}t\cos\omega_{IF}t \mp \hat{m}(t)\sin^2\omega_{IF}t$

$= \frac{m(t)}{2}\sin 2\omega_{IF}t \mp \frac{\hat{m}(t)}{2}\left(1 - \cos 2\omega_{IF}t\right)$

(G) $v_G(t) = \mp\frac{\hat{m}(t)}{2}$

(H) $v_H(t) = \pm\frac{m(t)}{2}$

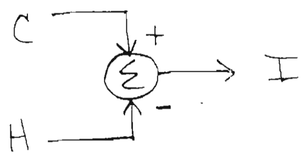
(I) $v_I(t) = v_C(t) + v_H(t)$

$= \frac{m(t)}{2} \pm \frac{m(t)}{2}$

$$v_I(t) = \begin{cases} m(t), & \text{USSB} \\ 0, & \text{LSSB} \end{cases}$$

EE 443 HW # page 2

5.18 Cont'd) To receive LSSB signals, <u>subtract</u> $v_H(t)$ from $v_c(t)$ at the summer.




(b.) see part (a.)

(c.) see part (a.)

(d.) IF should be centered at $f_c \pm 1.5$ KHz, (USSB / LSSB) have 3KHz BW and as small a roll-off factor as is economically feasible.

LPF should have 3KHz BW and as small a roll-off factor as is feasible, also.

√ 5-20 (a.) <u>0%</u> AM

(b.) $P_{norm} = A_c^2/2 = 10^2/2 = \underline{50W}$

(c.) $\Delta\theta_{max} = \underline{10 \text{ radians}}$

(d.) $\omega_d(t) = \frac{d\theta(t)}{dt} = -10(2000\pi)\sin(2000\pi t)$

$\Delta F = \frac{\Delta\omega}{2\pi} = \frac{10(2000\pi)}{2\pi} = 10^4 = \underline{10 \text{ kHz}}$

EE 443 HW # page 3

**5-22** $m(t) = A_m \cos(2\pi f_m t) = 4\cos(2\pi \times 10^3 t)$

(a.) $f_i(t) = f_c + \Delta F \cos(2\pi \times 10^3 t)$

$\Delta F = k_f A_m = \left(\frac{50\,Hz}{V}\right)(4V) = \underline{200\,Hz}$

(b.) $\beta = \frac{\Delta F}{f_m} = \frac{200\,Hz}{1\,KHz} = \underline{\underline{0.2}}$

**5-24.** (a.) $f_{BPF} = \frac{103.7}{8}\,mHz = \underline{\underline{12.96\,mHz}}$

$\Delta F_{BPF} = \frac{75\,KHz}{8} = 9.375\,KHz$

$BW_{BPF} = 2(\Delta F + f_m) = 2(9.375 + 15)\,KHz$

$= \underline{\underline{48.75\,KHz}}$

(b.) $f_{BPF} = f_c + f_o \Rightarrow f_o = 12.96 - 5 = \underline{\underline{7.96\,mHz}}$

$f_{BPF} = f_c - f_o \Rightarrow f_o = 12.96 + 5 = \underline{\underline{17.96\,mHz}}$

$\uparrow f_c = 5\,mHz$

(c.) $\Delta F_{FME} = \frac{75\,KHz}{8} = \underline{\underline{9.38\,KHz}}$

EE 443 Hw # page 4

**5-26.** (a.) $\Theta(t) = D_p m_p(t) = 20\cos\omega_1 t$

$\Rightarrow m_p(t) = \frac{20}{D_p}\cos\omega_1 t = \underline{\underline{0.2\cos(2000\pi t)}}$

$m_p(t)_{peak} = \underline{\underline{0.2\text{ V}}}$ ; $f_m = \underline{\underline{1\text{ KHz}}}$

(b.) $\Theta(t) = D_f\int_{-\infty}^{t} m_f(\lambda)\,d\lambda = 20\cos\omega_1 t$

$\Rightarrow m_f(t) = \frac{20}{D_f}\frac{d}{dt}\left[\cos\omega_1 t\right]$

$= \frac{-20}{10^6}(2000\pi)\sin\omega_1 t$

$m_f(t) = \underline{\underline{-.1257\sin\omega_1 t}}$

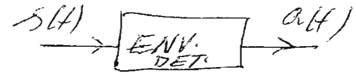
$m_f(t)_{peak} = \underline{\underline{.1257\text{ V}}}$ ; $f_m = \underline{\underline{1\text{ KHz}}}$

(c.) $P_{AV} = \frac{V_{rms}^2}{R} = \frac{(500)^2}{2(50)} = \underline{\underline{2.5\text{ KW}}}$

$PEP = \underline{\underline{2.5\text{ KW}}}$

EE 443 Hw # 8 Chapt. 3 page 5

3.24)




$$s(t) = m(t)\cos 2\pi f_c t + A_c \cos(2\pi f_c t + \varphi)$$

$$= m(t)\cos 2\pi f_c t + A_c \cos\phi \cos 2\pi f_c t - A_c \sin\phi \sin 2\pi f_c t$$

$$= \underbrace{(m(t) + A_c\cos\varphi)}_{\text{In phase comp.}} \cos 2\pi f_c t - \underbrace{A_c \sin\phi}_{\text{quadrature component}} \sin 2\pi f_c t$$

$$a(t) = \sqrt{(m(t) + A_c\cos\phi)^2 + A_c^2 \sin^2\phi}$$

$$= \sqrt{m^2(t) + A_c^2\cos^2\phi + 2A_c\cos\phi\, m(t) + A_c^2\sin^2\phi}$$

$$= \sqrt{m^2(t) + A_c^2 + 2A_c\cos\phi\, m(t)} \qquad (1)$$

a) If $\varphi = 0 \Rightarrow a(t) = \sqrt{m^2(t) + A_c^2 + 2A_c m(t)} = \sqrt{(m(t)+A_c)^2}$

$\Rightarrow a(t) = |m(t) + A_c| = m(t) + A_c$ if $|m(t)| < A_c$

b) For $\phi \neq 0$ and $|m(t)| \ll A_c/2$ using eq (1)

$$a(t) = A_c\sqrt{1 + \frac{2}{A_c}\cos\phi\, m(t) + \frac{m^2(t)}{A_c^2}}$$

$$\simeq A_c\sqrt{1 + \frac{2}{A_c}\cos\phi\, m(t)}$$

we can neglect $\frac{m^2(t)}{A_c^2}$ since $|m(t)| \ll A_c/2$

$$= A_c\left[1 + \frac{1}{2}\cdot\frac{2}{A_c}\cos\phi\, m(t)\right]$$

using $(1+x)^\alpha \simeq 1 + \alpha x$ if $x \ll 1$

$$= A_c + \cos\phi\, m(t)$$

## 3.11 HW 11

### Local contents



This was not collected. Practice problems for class only.

### 3.11.1 Problems

Find Handout, do 4 problems for sampling



high noise peaks. It is apparent that these false pulses have a finite though small probability of occurrence when the noise is Gaussian, no matter how small its standard deviation is compared with the peak amplitude of the pulses. As the transmission bandwidth is increased indefinitely, the accompanying increase in average noise power eventually causes the false pulses to occur often enough, thereby causing loss of the wanted message signal at the receiver output. *We thus find, in practice, that both PPM and PDM systems suffer from a threshold effect similar to that experienced in FM systems.*

## Synchronization in Pulse-Time Modulation Systems

As with PAM systems, synchronization in pulse-time modulation systems is established by transmitting a distinctive marker per frame. In a PDM system, the marker may be identified by omitting a pulse, as illustrated in Fig. 7.13(*c*) for a PDM system involving three independent message sources. One method of identifying such a marker in the receiver is to utilize the charging time of a simple resistor-capacitor circuit to measure the duration of the intervals between duration-modulated pulses. The time constant of the circuit is chosen so that, during a marker interval, the voltage across the capacitor rises to a value considerably higher than that during the normal charging interval. Thus, by applying the output of the circuit to a slicer with an appropriate slicing level, the presence of a marker is detected.

In a PPM system, the marker pulse may be identified by making its duration several times longer than that of the message pulses, as illustrated in Fig. 7.13(*d*). At the receiver, the marker pulses may be separated from the message pulses by using a procedure essentially similar to that described for the PDM system. In this case, however, the capacitor is charged during the time of occurrence of each pulse, and discharged during the intervening intervals. Accordingly, the voltage across the capacitor reaches its highest value during the presence of a marker pulse, and the marker pulses are thereby separated from the message pulses.

## Problems

✓ ***Problem 7.1*** The signal

$$g(t) = 10\cos(20\pi t)\cos(200\pi t)$$

is sampled at the rate of 250 samples per second.

(a) Determine the spectrum of the resulting sampled signal.
(b) Specify the cutoff frequency of the ideal reconstruction filter so as to recover $g(t)$ from its sampled version.
(c) What is the Nyquist rate for $g(t)$?
(d) By treating $g(t)$ as a band-pass signal, determine the lowest permissible sampling rate for this signal.

√ ***Problem 7.2*** The signals

$$g_1(t) = 10\cos(100\pi t)$$

and

$$g_2(t) = 10\cos(50\pi t)$$

are both sampled at the rate of 75 samples per second. Show that the two sequences of samples thus obtained are identical. What is the reason for this phenomenon?

√ ***Problem 7.3*** The signal

$$g(t) = 10\cos(60\pi t)\cos^2(160\pi t)$$

is sampled at the rate of 400 samples per second. Determine the range of permissible cutoff frequencies for the ideal reconstruction filter that may be used to recover $g(t)$ from its sampled version.

√ ***Problem 7.4*** A signal $g(t)$ consists of two frequency components $f_1 = 3.9$ kHz and $f_2 = 4.1$ kHz in such a relationship that they just cancel each other out when the signal $g(t)$ is sampled at the instants $t = 0, T, 2T, \ldots$, where $T = 125\ \mu s$. The signal $g(t)$ is defined by

$$g(t) = \cos\left(2\pi f_1 t + \frac{\pi}{2}\right) + A\cos(2\pi f_2 t + \phi)$$

Find the values of amplitude $A$ and phase $\phi$ of the second frequency component.

***Problem 7.5*** Let $E$ denote the energy of a strictly band-limited signal $g(t)$. Show that $E$ may be expressed in terms of the sample values of $g(t)$, taken at the Nyquist rate, as follows

$$E = \frac{1}{2W}\sum_{n=-\infty}^{\infty}\left|g\left(\frac{n}{2W}\right)\right|^2$$

where $W$ is the highest frequency component of $g(t)$.

***Problem 7.6*** The spectrum of a signal $g(t)$ is shown in Fig. P7.1. This signal is sampled at the Nyquist rate with a periodic train of rectangular pulses of duration 50/3 milliseconds. Plot the spectrum of the sampled signal for frequencies up to 50 hertz.

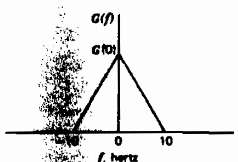



**Figure P7.1**

***Problem 7.7*** This problem is aimed at investigating the fact that practical electronic switching circuits will not produce a sampling function that consists of exactly rectangular pulses. Let $h(t)$ denote some arbitrary pulse shape, so that the sampling function $c(t)$ may be expressed as



$$c(t) = \sum_{n=-\infty}^{\infty} h(t - nT_s)$$

where $T_s$ is the sampling period. The sampled version of an incoming analog signal $g(t)$ is defined by

$$s(t) = c(t)g(t)$$

(a) Show that the Fourier transform of $s(t)$ is given by

$$S(f) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} G\left(f - \frac{n}{T_s}\right)H\left(\frac{n}{T_s}\right)$$

where $G(f) = F[g(t)]$ and $H(f) = F[h(t)]$.

(b) What is the effect of using the arbitrary pulse shape $h(t)$?

***Problem 7.8*** Consider a continuous-time signal $g(t)$ of finite energy, with a continuous spectrum $G(f)$. Assume that $G(f)$ is sampled uniformly at the discrete frequencies $f = kF_s$, thereby obtaining the sequence of frequency samples $G(kF_s)$, where $k$ is an integer in the entire range $-\infty < k < \infty$, and $F_s$ is the frequency sampling interval. Show that if $g(t)$ is duration-limited, so that it is zero outside the interval $-T \leq t \leq T$, then the signal is completely defined by specifying $G(f)$ at frequencies spaced $1/2T$ hertz apart.

***Problem 7.9***

(a) Consider a stationary process $X(t)$ that is *not* strictly band-limited in the band $W$; that is,

$$S_X(f) \neq 0, \quad |f| > W$$

where $S_X(f)$ is the power spectral density of the process. The process $X(t)$ is applied to an ideal low-pass filter defined by the transfer function

$$H(f) = \begin{cases} 1, & |f| < W \\ 0, & |f| > W \end{cases}$$

producing the process $X_f(t)$. This process is next sampled at a rate equal to $2W$, producing the sequence of samples $X_f(n/2W)$. An approximate reconstruction of the original process is defined by

$$Y(t) = \sum_{n=-\infty}^{\infty} X_f\left(\frac{n}{2W}\right)\text{sinc}\left[2W\left(t - \frac{n}{2W}\right)\right]$$

Show that the mean-square value of the *sampling error* is

$$\mathscr{E} = E[(X(t) - Y(t))^2] = 2\int_W^{\infty} S_X(f)\,df$$

(b) Given that

$$S_X(f) = \frac{f_0}{f^2 + f_0^2}$$

determine the corresponding value of the mean-square error $\mathscr{E}$, and plot it as a function of $W/f_0$.

***Problem 7.10*** Consider a sequence of samples $x(nT_s)$ obtained by sampling a continuous-time signal $x(t)$ at the rate $1/T_s$. It is required to increase the sampling period $T_s$ to a new value

### 3.11.2 Key solution

EE 443 Chap 7 page 1

**Chapter 7**

**Pulse-Analog Modulation**

**Problem 7.1**

(a) The signal g(t) is

$$g(t) = 10\cos(20\pi t)\cos(200\pi t)$$

$$= 5[\cos(220\pi t) + \cos(180\pi t)]$$

The Fourier transform of g(t) is

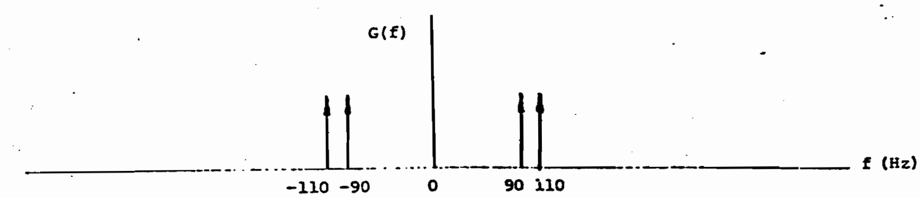
$$G(f) = 2.5[\delta(f-110) + \delta(f+110) + \delta(f-90) + \delta(f+90)]$$

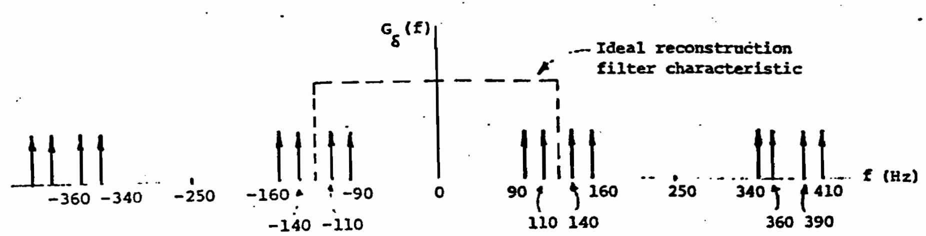
Hence, the spectrum of the sampled version of g(t), with a sampling period $T_s$ = 1/250 s, is given by

$$G_s(f) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} G\left(f - \frac{n}{T_s}\right) = f_s \sum_{m=-\infty}^{+\infty} G(f - m f_s)$$

$$= 250 \times 2.5 \sum_{n=-\infty}^{\infty} [\delta(f-110-250n) + \delta(f+110-250n) + \delta(f-90-250n) + \delta(f+90-250n)]$$

(b) The spectra G(f) and $G_\delta(f)$ are illustrated below:



EE 449 Chapter #1 . page 2

From this diagram, we deduce that in order to recover the original signal g(t) from $g_\delta(t)$, we need to use a low-pass filter with a cutoff frequency that is greater than 110 Hz but less than 140 Hz.

(c) The highest frequency component of g(t) is 110 Hz. The Nyquist rate of g(t) is therefore 220 Hz.

(d) The signal g(t) may be viewed as a band-pass signal occupying the frequency interval 90 to 110 Hz, that is,

$$f_u = 110 \qquad W = 110 - 90 = 20$$

$$f_s = \frac{2 f_u}{m}$$

$$m \leq \frac{f_u}{W} = \frac{110}{20} = 5.5 \implies m = 5$$

$$f_s = \frac{2 f_u}{m} = \frac{2 \times 110}{5} = 44 Hz.$$

**Problem 7.2**

The spectrum of $g_1(t)$ is

$$G_1(f) = 5[\delta(f-50) + \delta(f+50)]$$

Hence, the spectrum of the sampled version of $g_1(t)$, using a sampling period $T_s = 1/75$ s, is

$$G_{1\delta}(f) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} G_1(f - \frac{n}{T_s})$$

$$= 375 \sum_{n=-\infty}^{\infty} [\delta(f-50-75n) + \delta(f+50-75n)] \quad (1)$$

Next, the spectrum of $g_2(t)$ is

$$G_2(f) = 5[\delta(f-25) + \delta(f+25)]$$

Hence, the spectrum of the sample version of $g_2(t)$, using a sampling period $T_s$ =1/75 s, is

$$G_{2\delta}(f) = 375 \sum_{n=-\infty}^{\infty} [\delta(f-25-75n) + \delta(f+25-75n)] \quad (2)$$

In the right-hand side of Eq. (2), substitute n = ℓ-1 for the first term, and n = m+1 for the second term, and so rewrite this equation as follows:

EE 443 Chapt #7 page 3

$$G_{2\delta}(f) = 375 \sum_{\ell=-\infty}^{\infty} \delta(f+50-75\ell) + 375 \sum_{m=-\infty}^{\infty} \delta(f-50-75m)$$

$$= 375 \sum_{n=-\infty}^{\infty} [\delta(f-50-75n) + \delta(f+50-75n)] \qquad (3)$$

We thus find from Eqs. (1) and (3) that the spectra $G_{1\delta}(f)$ and $G_{2\delta}(f)$ are identical. That is, the sample versions of $g_1(t)$ and $g_2(t)$ are identical.

We note that the Nyquist rate of $g_1(t)$ is 100 Hz; hence, with a sampling rate of 75 Hz, the signal $g_1(t)$ is under-sampled by 25 Hz below the Nyquist rate. On the other hand, the Nyquist rate of $g_2(t)$ is 50 Hz; hence, the signal $g_2(t)$ is over-sampled by 25 Hz above the Nyquist rate. Thus, although $g_1(t)$ and $g_2(t)$ represent two sinusoidal waves of different frequencies, we find that by under-sampling $g_1(t)$ and over-sampling $g_2(t)$ appropriately, their sampled versions are identical.

Problem 7.3

Express the signal g(t) as

$$g(t) = 10 \cos(60\pi t) \cos^2(160\pi t)$$

$$= 5 \cos(60\pi t)[1 + \cos(320\pi t)]$$

$$= 5 \cos(60\pi t) + 2.5 \cos(380\pi t) + 2.5 \cos(260\pi t)$$

The spectrum of g(t) is

$$G(f) = 2.5[\delta(f-30) + \delta(f+30)] + 1.25[\delta(f-190) + \delta(f+190)] + 1.25[\delta(f-130) + \delta(f+130)]$$

The corresponding spectrum of the sampled versionof g(t), using a sampling rate of 400 Hz, is therefore

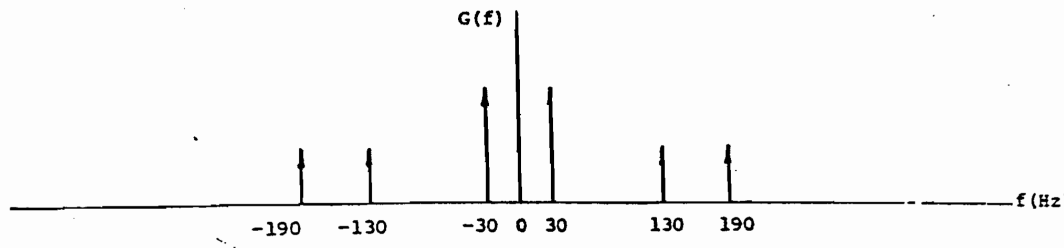
$$G_\delta(f) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} G(f - \frac{n}{T_s})$$

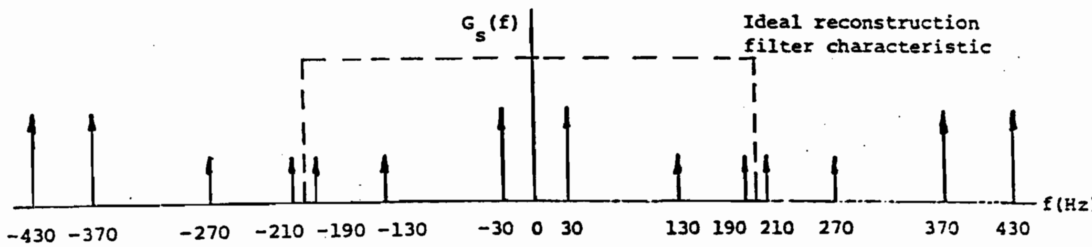
$$= 400 \sum_{n=-\infty}^{\infty} \left[ \begin{array}{l} 2.5[\delta(f-30-400n) + \delta(f+30-400n)] \\ + 1.25[\delta(f-190-400n) + \delta(f+190-400n)] \\ + 1.25[\delta(f-130-400n) + \delta(f+130-400n)] \end{array} \right]$$

The spectra G(f) and $G_\delta(f)$ are illustrated below:

EE 443 Chapt. #7 . page 4

[redacted]




From this diagram, we deduce that in order to recover the original signal g(t) from its sampled version, the low-pass reconstruction filter must have a cutoff frequency greater than 190 Hz but less than 210 Hz.

Problem 7.4

The signal at the sampling instants is:

$$g(nT) = \cos(2\pi f_1 nT + \frac{\pi}{2}) + A\cos(2\pi f_2 nT + \phi)$$

$$= 0\,, \qquad n = 0, 1, 2, \ldots$$

At n = 0,

$$\cos(\frac{\pi}{2}) + A\cos\phi = 0\,. \tag{1}$$

At n = 1, 2, ..., with $f_1$ = 3.9 kHz, $f_2$ = 4.1 kHz, and T = 125 μs, we have

$$\cos(0.975n\pi + \frac{\pi}{2}) + A\cos(1.025\,n\pi + \phi) = 0. \tag{2}$$

From (2) and $\cos(0.975n\pi + \frac{\pi}{2})$ being non-zero, A must be non-zero. From (1) and A being non-zero, $\phi$ must be $\pm\frac{\pi}{2}$. Equation (2) then becomes:

$$-\sin(0.975n\pi) \pm A\sin(1.025n\pi) = 0. \tag{3}$$

Since sin(•) is odd symmetric about $n\pi$, A equals 1 and the ambiguous sign in (3) is negative. Therefore, $\phi = \frac{\pi}{2}$.